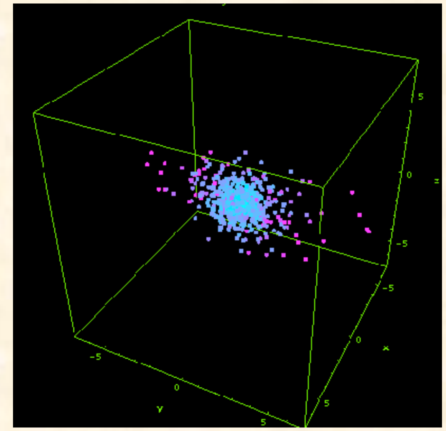

The N-body Problem

Luis A. Aguilar

Instituto de Astronomía/UNAM

México

The *N*-body problem is, perhaps, the oldest and most fruitful unsolved problem in the history of science.

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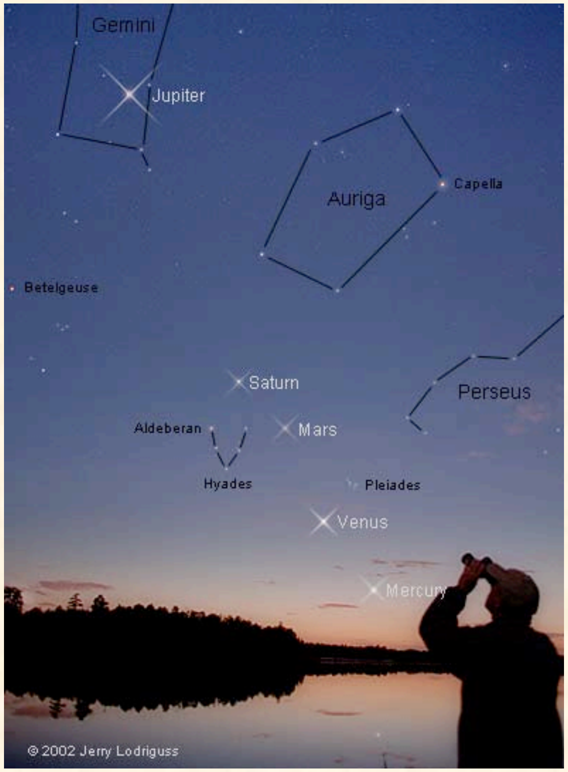
Its origin goes back to mankind's necessity to measure the pass of time to anticipate animal migrations and, later on, follow agricultural cycles.

Ancient people found their first calendar in the heavens.



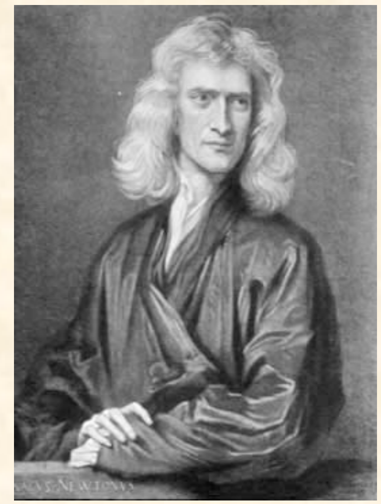
The *N*-body problem is, perhaps, the oldest and most fruitful unsolved problem in the history of science.

To read the calendar in the sky, early astronomers developed empirical models based on the regular patterns followed by planets and stars.

Without knowing it, they were developing models to solve an *N*-body problem: The Solar System.

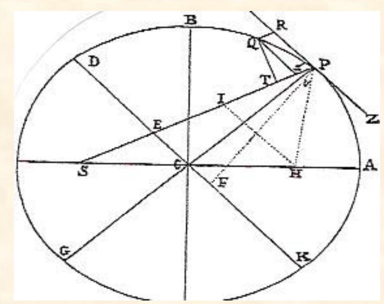
Isaac Newton and the *N*-body problem

In the XVII century, Isaac Newton formulated the classic form of the gravitational interaction. Except for the law of specular reflection and Arquimedes principle, this was the first mathematical description of a natural phenomenon.



“Planetas omnes in se muto graves esse jam ante probavimus,
ut & gravitatem in unumquemque seorsim spectatum
esse reciproce ut quadratum distantiœ locorum a centro planetæ.
Et inde consequens est gravitatem in omnes proportionalem
esse materiæ in iisdem”.

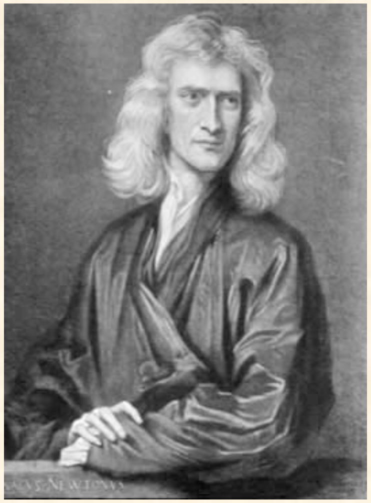
Isaac Newton,
Proposición vii, Teorema VII.
Philosophiæ Naturalis Principia Mathematica.



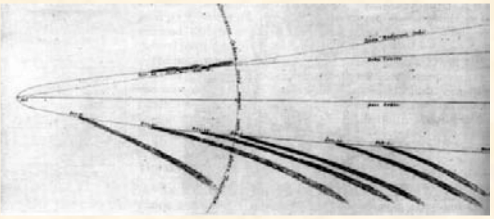
Isaac Newton and the *N*-body problem

The solution of the two-body problem, lead Newton to the invention of Calculus. The three-body problem, however, proved intractable. Newton wrote that its solution,

> "Unless I am much mistaken, it would exceed the force of human wit to consider so many causes of motion at the same time".



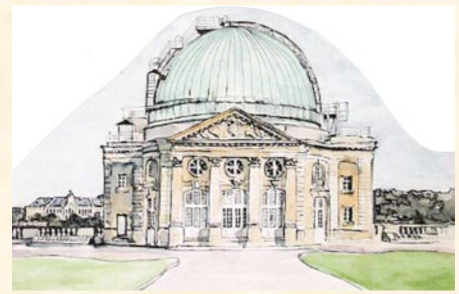
Nevertheless, the two-body problem was sufficient to predict the return of a comet when the influence of other planets was included as a perturbation: Halley's comet.



The *N*-body problem and the quest for power among European powers

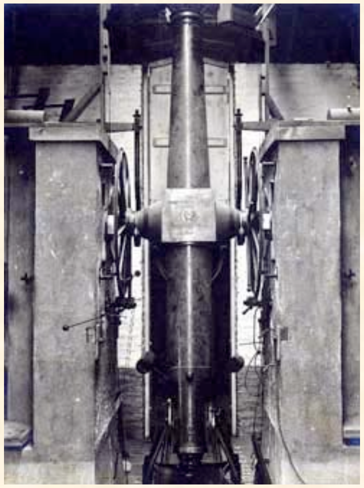
The great European powers established national observatories and promoted the development of mathematics and astronomy, trying to solve a problem of large strategic importance: navigation.

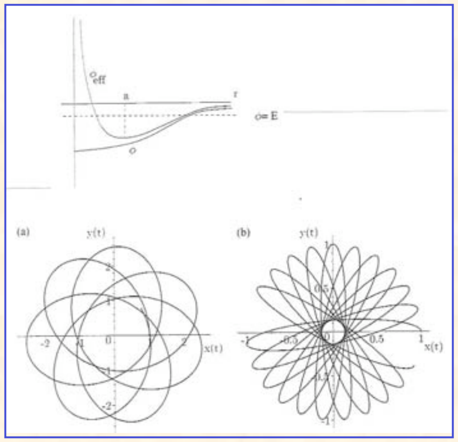
The search for a solution split in two camps:
Some turned to mechanical devices to measure time accurately and thus determine geographical position, while others kept looking at the sky for the perfect clock.



The *N*-body problem and the development of modern science

The second approach motivated the study of the *N*-body problem and brought an enormous development of Mathematics and the Physical Sciences.





The *N*-body problem and the development of modern science

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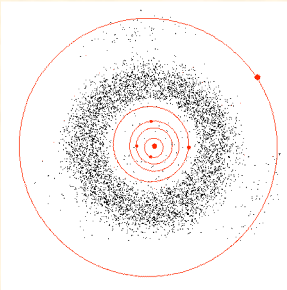

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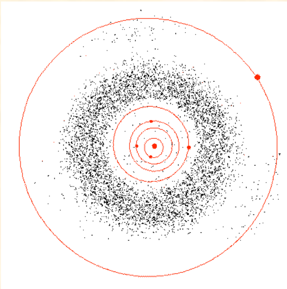

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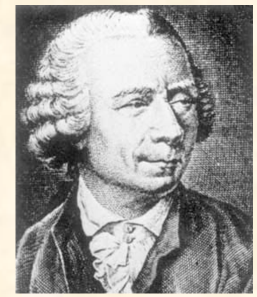
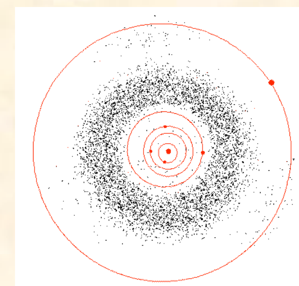
• Henri Jules Poincaré studied very thoroughly the 3-body problem. The methods he introduced to study its stability lead to the development of Topology and to the discovery of the concept of Chaos.



The *N*-body problem and the development of modern science

The study of the *N*-body problem followed two paths: the first relied on finding approximations to the true solution. This gave rise to *Celestial Mechanics*. The second approach looked for solutions transforming the original problem in another equivalent. This developed into *Analytic Mechanics* or *Rational Mechanics*.

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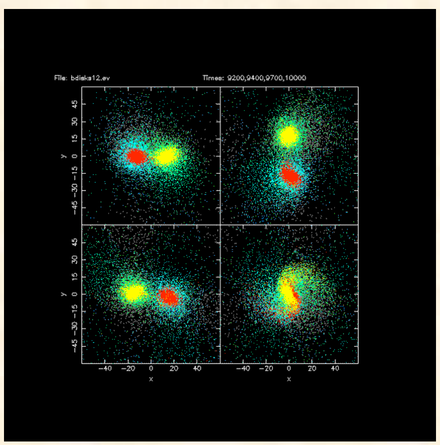
In the last 100 years, the study of the *N*-body problem lead to the development of another area of inquiry: *Stellar Dynamics*.



The *N*-body problem is of great interest for several sciences

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(i) $M = T^*W$ con estructura canónica simpléctica,

$$W = \mathbf{R}^3 \times \mathbf{R}^3 \setminus \Delta, \qquad \Delta = \{(\mathbf{q}, \mathbf{q}) \mid \mathbf{q} \in \mathbf{R}^3\};$$

(ii) $m \in M$ (condiciones iniciales);

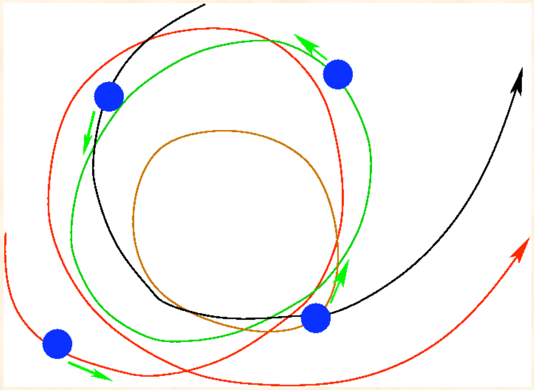
(iii) $\mu \in \mathbf{R}$, $\mu > 0$ (cociente de masas);

(iv) $H^\mu \in F(M)$ dado por

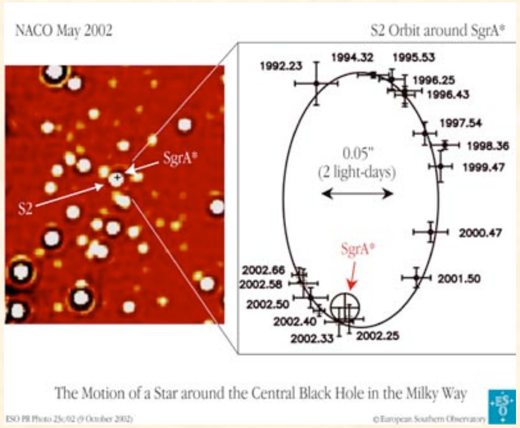
$$H^\mu(\mathbf{q}, \mathbf{q}', \mathbf{p}, \mathbf{p}') = \frac{\|\mathbf{p}\|^2}{2\mu} + \frac{\|\mathbf{p}'\|^2}{2\mu} - \frac{1}{\|\mathbf{q} - \mathbf{q}'\|},$$

donde $\mathbf{q}, \mathbf{q}' \in \mathbf{R}^3$, $\mathbf{p}, \mathbf{p}' \in (\mathbf{R}^3)^*$, y donde $\|\ \|$ es la norma euclidiana en $\mathbf{R}^3$.

Mathematics



Physics



Astronomy

```
do i = 1, N
  F_i = 0
end do

do i = 1, N - 1
do j = 1 + 1, N
  F_ij = (G m_i m_j / r_ij^3) r_ij
  F_i = F_i - F_ij
  F_j = F_j - F_ij
end do
end do

do i = 1, N
  v_i(t + Δt) = v_i(t) + (F_i / m_i) Δt
  r_i(t + Δt) = r_i(t) + v_i Δt
end do
```

Computer Science

The *N*-body problem in Physics

The *N*-body problem in Physics

The gravitational *N*-body problem (in the classical regime), has a historical development that is inverted with respect to other branches of Physics: the rest has started from a global description of phenomena to develop, subsequently, a local description of the fundamental interaction among the components of the system.

Thermodynamics ➜ *Statistical Mechanics,*
Fluid Mechanics ➜ *Kinetic Theory of gases,*
Bulk properties of materials ➜ *Atomic and molecular structure.*

The *N*-body problem in Physics

In the gravitational problem we knew the interaction rule from the beginning:

$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$

The *N*-body problem in Physics

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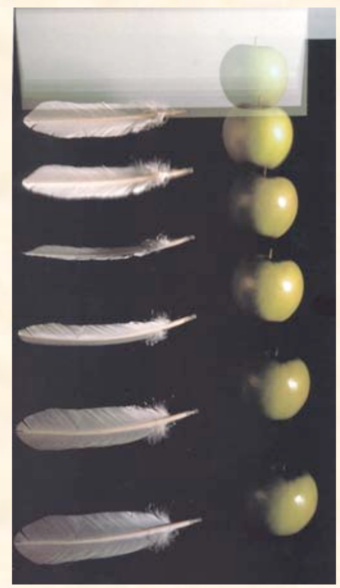
This interaction is unique in the Universe. No other force affects everything in it, whether it is matter or energy, apples or galaxies. Its effect does not vanish with distance, only diminished. Although the weakest of all known forces, it is gravity that truly shapes our Universe.



The *N*-body problem in Physics

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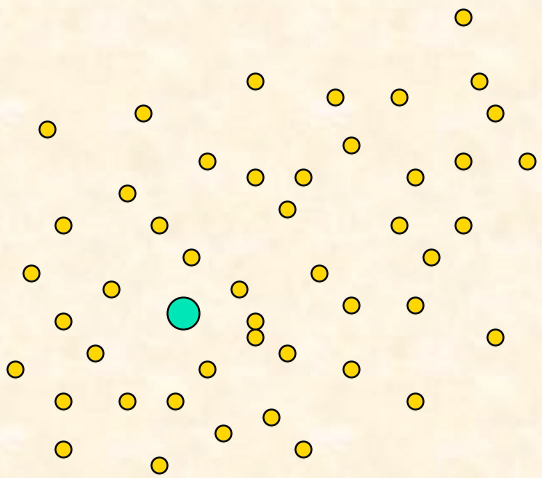
There are two very important features of gravity that we will discuss now: the fact that it is a *long range force* and that it has *no characteristic scale*.

The *N*-body problem in Physics

Let's imagine that we inhabit a planet at the center of a spherical galaxy that contains a huge number of stars.

It is clear that the net average force on our planet will be zero, since the distribution of matter is symmetric around us: there is, on average, the same number of stars pulling us on each direction.



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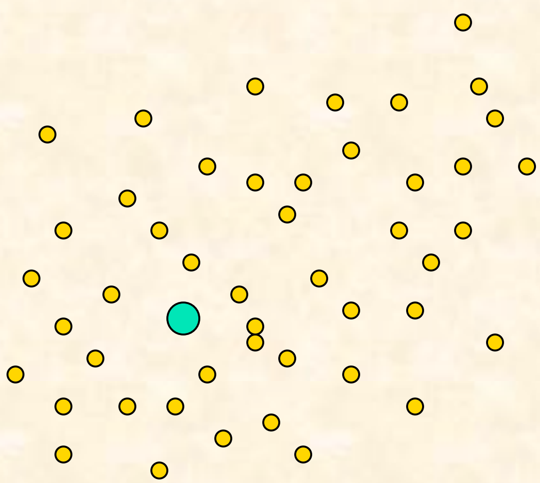
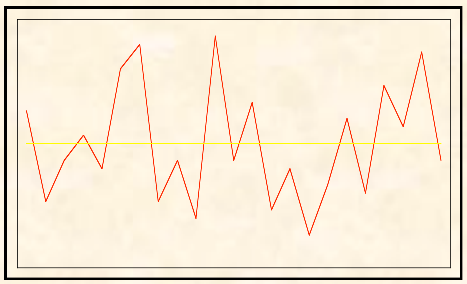
The force, however, will fluctuate around zero, due to the discrete nature of the mass distribution.

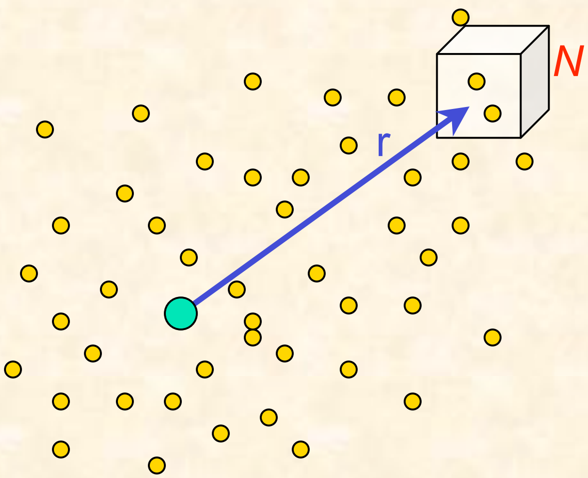




The *N*-body problem in Physics

If we center our attention to a particular region which contains N stars at a distance r from us, it is clear that the force due to it is proportional to:

$$\boxed{F \propto \frac{N}{r^2}}$$



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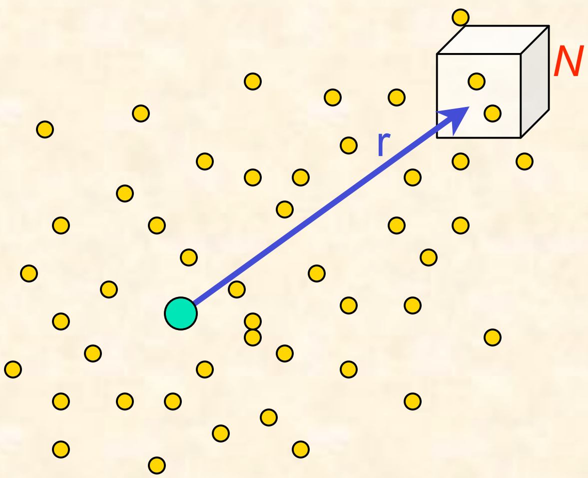
$$\boxed{\Delta F \propto \frac{\Delta N}{r^2} = \frac{\sqrt{N}}{r^2}}$$



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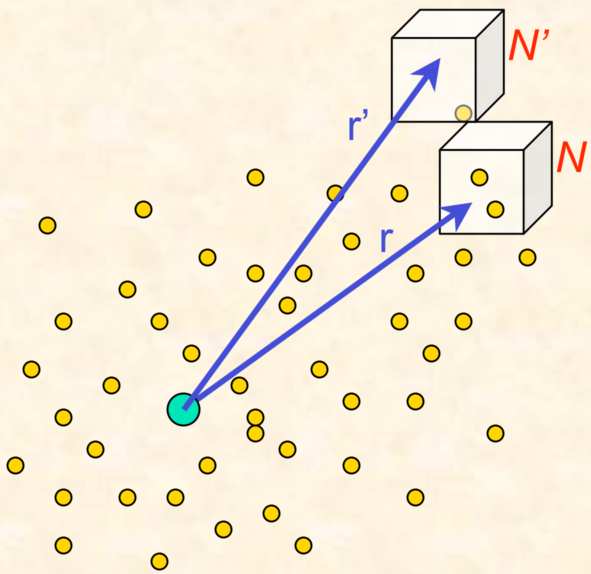


If we now look at another region at a distance r' and ask that it exerts the same force on us, it is clear that it must contain a number of stars given by:

$$\boxed{N' \propto r'^2}$$

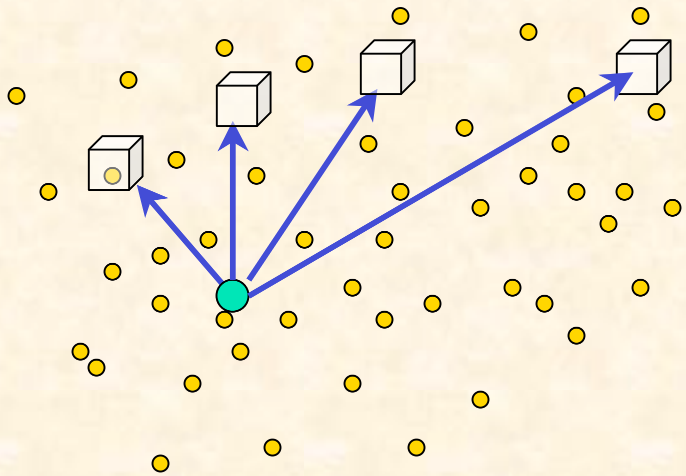
And its fluctuations will be given by,

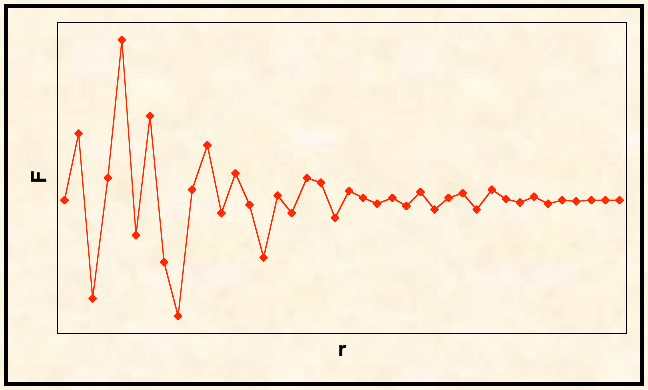
$$\boxed{\Delta F' \propto \frac{\sqrt{N'}}{r^2} \propto \frac{1}{r'}}$$

The *N*-body problem in Physics

$$\boxed{\Delta F \propto \frac{1}{r}}$$

This is a very important result, it means that if we split a galaxy into parcels, such that each one exerts on average the same force on us, the fluctuations in their forces will shrink with distance as *1/r*.





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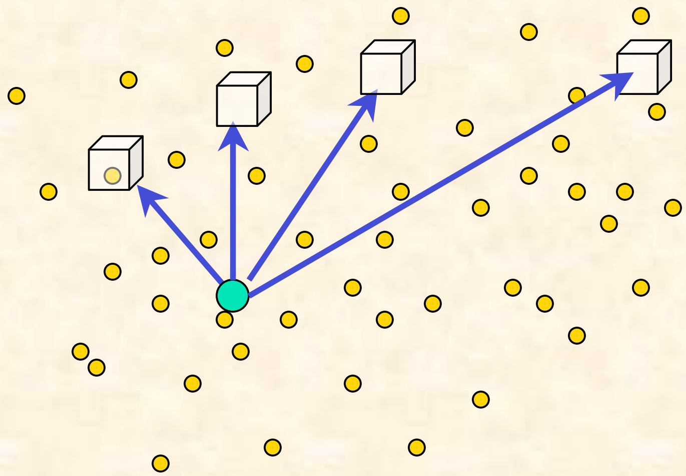
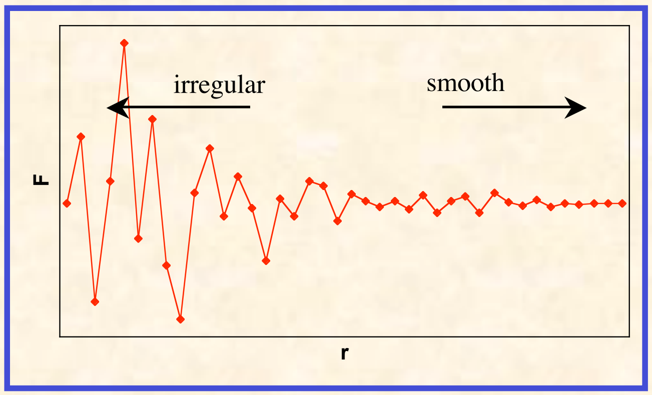
We can then split the gravitational influence into two asymptotic regimes: a fluctuating part due to close neighbors and a smooth part due to distant stars.





The *N*-body problem in Physics

The characteristic time in which the irregular part dominates is called the *collisional time* and it can be shown that it is of the form:

$$\boxed{t_{col} \propto \left(\frac{R}{\sqrt{<v^2>}}\right) N \log(N)}$$

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On the other hand, the natural time unit in a gravitational system is the so called *dynamical time* in which particles cross the system:

$$\boxed{t_{din} \propto \left(\frac{R}{<v^2>}\right)}$$

Their ratio measures the importance of the irregular part with respect to the smooth one:

$$\boxed{t_{col} / t_{din} \propto N / \log(N)}$$

The *N*-body problem in Physics

The smaller this ratio, the more the system will be dominated by the irregular regime, where the force is dominated by collisions with close neighbors. Thus, we can split self-gravitating *N*-body systems in two types: *collisional* and *collisionless systems*; the fundamental difference between them being set by the number of particles.

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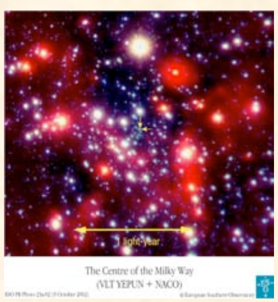
System	N	t_{col} (10^6 years)	t_{din} (10^6 years)	t_{col}/t_{din}
Stelar group	≤ ~10	≤1	≤1	~1
Globular cluster	~10^6	≤10^3	~10	~10^2
Galaxy	~10^{11}	~10^7	10^2	~10^5
Cluster of galaxies	10^2-10^3	~10^3	10^3	~10



The *N*-body problem in Physics

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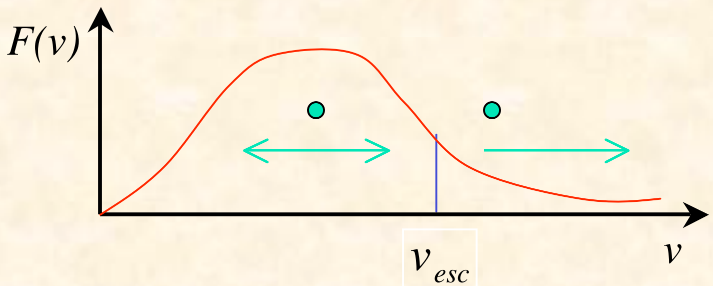

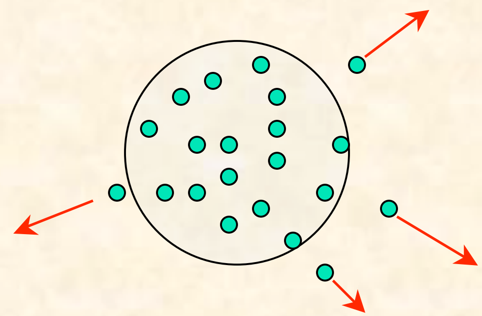

However, all self-gravitating systems of finite mass have a finite escape velocity. Since the Maxwell-Boltzmann distribution extends to infinite velocities, this means that within a collisional time, a fraction of the system escapes.

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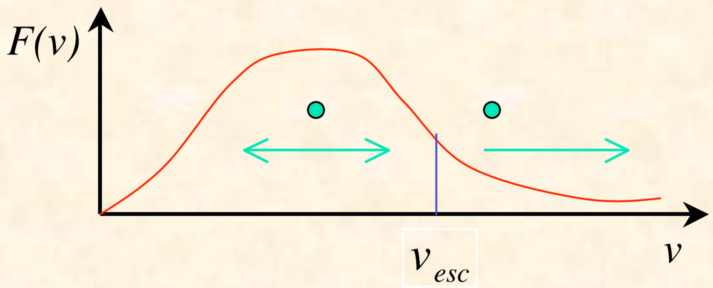

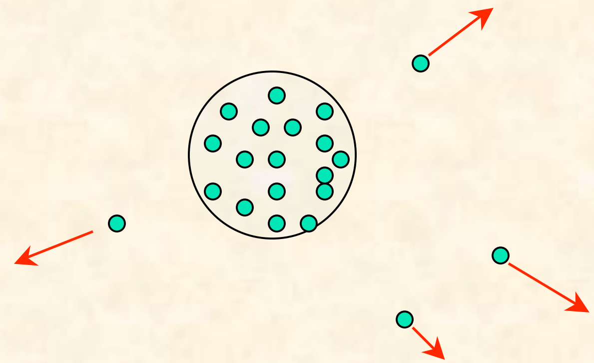

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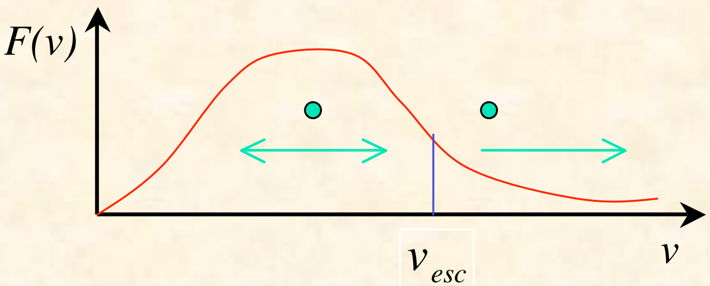

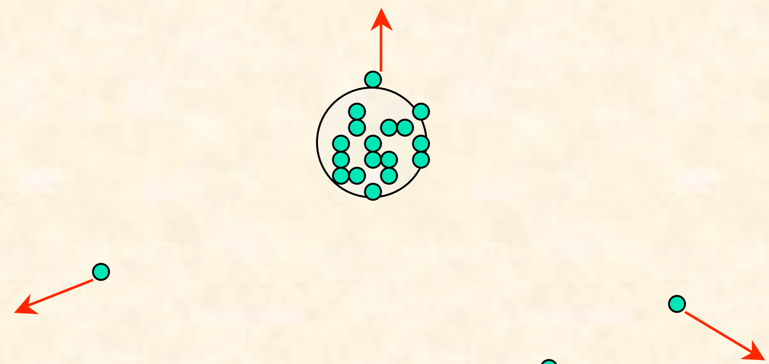

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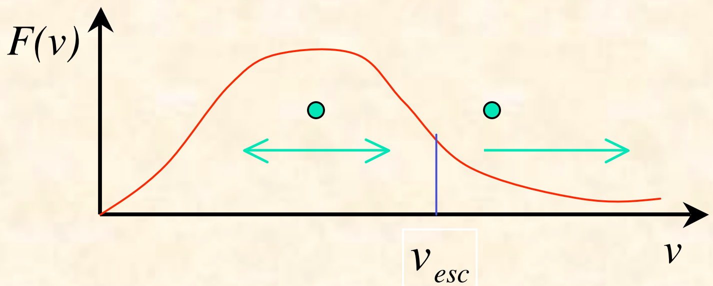

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The asymptotic state to which all self-gravitating systems converge is that of a central singularity surrounded by a halo of particles that escapes to infinity.

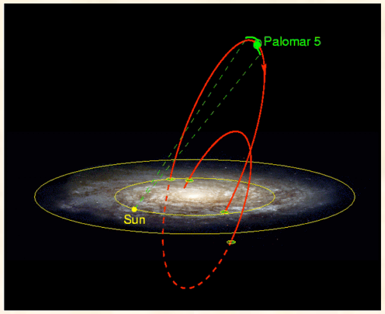
The *N*-body problem in Astronomy

The *N*-body problem in Astronomy

Can this really occur in the Universe?

The *N*-body problem in Astronomy

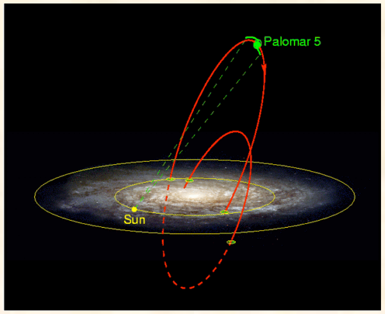
In the halo of our Galaxy there are spherical blobs formed by about a million stars each. These are *globular clusters*, and they are among the oldest components of the Galaxy.



As we have seen, the collisional time in these clusters is around 10^9 years, or about one tenth of their ages.

The *N*-body problem in Astronomy

In the halo of our Galaxy there are spherical blobs formed by about a million stars each. These are *globular clusters*, and they are among the oldest components of the Galaxy.



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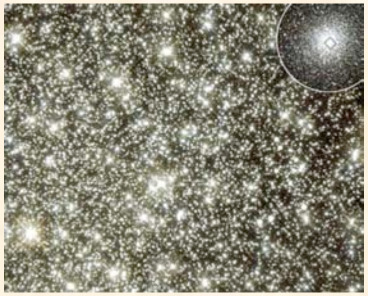
Globular clusters are obvious candidates to look for black holes.

The *N*-body problem in Astronomy

Up to now, the search for balck holes at the centers of globular clusters has not been succesful.



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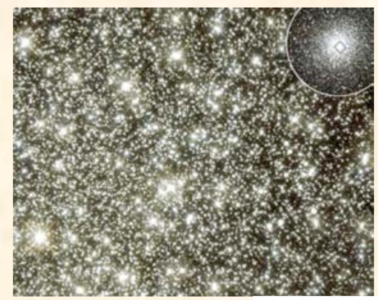
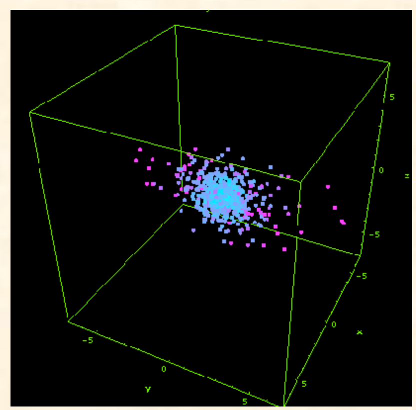
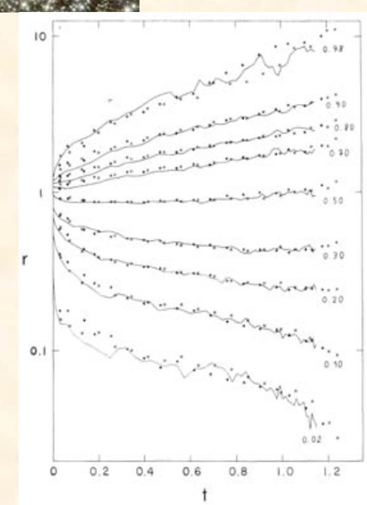


The numerical models, however, indicate that a central singularity must form.



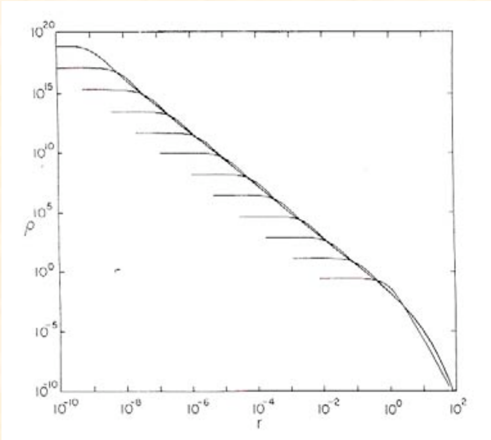


The *N*-body problem in Astronomy

What's going on then?

The *N*-body problem in Astronomy

According to theoretical models, the density at the center of the cluster increases by about, 18 orders of magnitude!



The N-body problem in Astronomy

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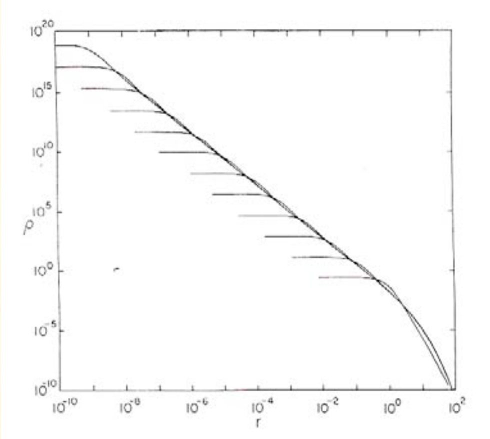
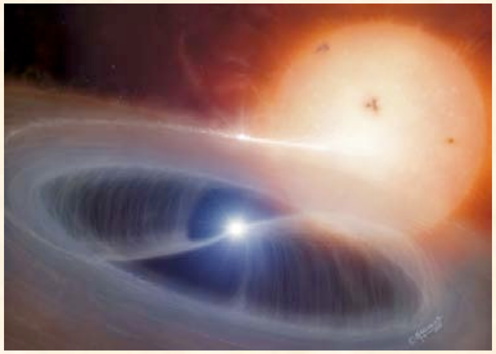
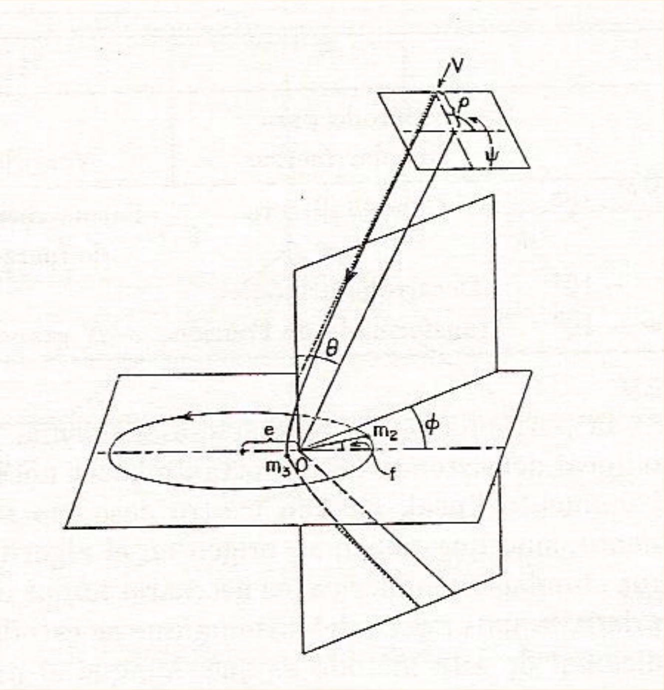




With such a tremendous increase, the mean stellar separations become comparable to the size of the Solar System. Such crowdness favors the formation of binary stellar systems.

The *N*-body problem in Astronomy

A binary system, contrary to a single star, can absorb or inject energy to its surroundings, by exchanging orbital energy with collisional kinetic energy when encountering a third star.



The *N*-body problem in Astronomy

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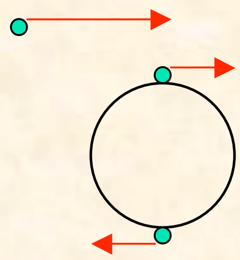

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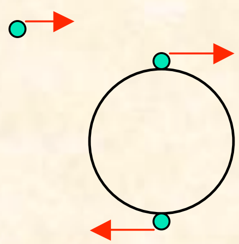

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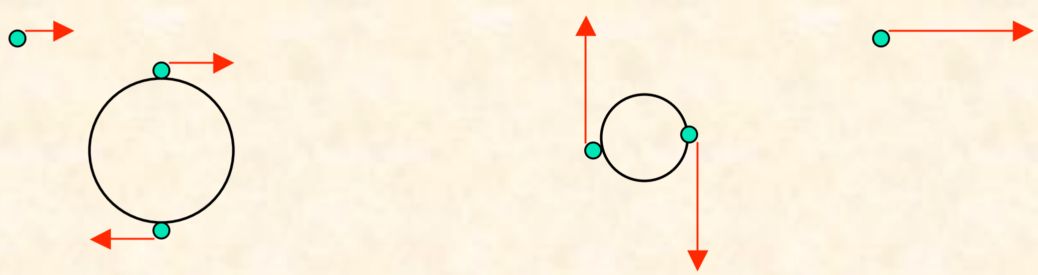
Losing energy, the binary shrinks and increases its kinetic energy even more.



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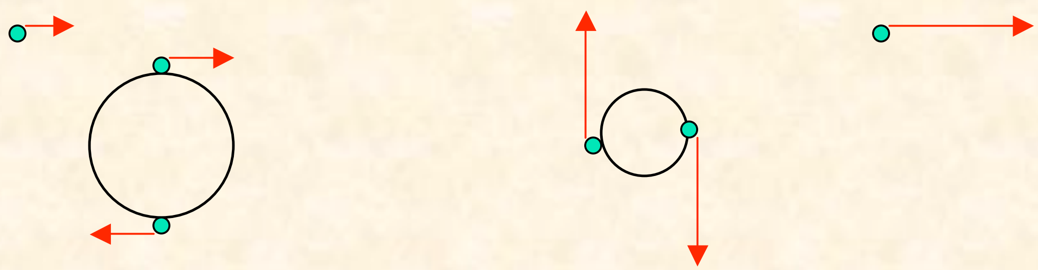
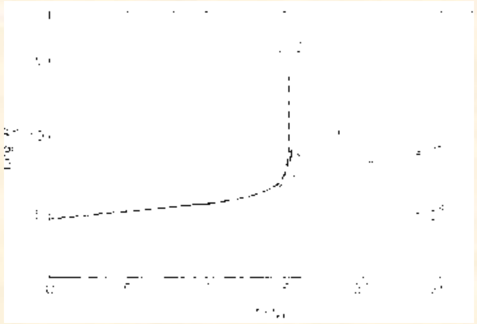


This gives rise to a dichotomy: soft binaries become even softer and disappear, while hard binaries become harder.

The *N*-body problem in Astronomy

This behavior is very important for the evolution of globular clusters: as the nucleus contracts and the central density increases, binary star systems are created, and the hardest of them dominates the local dynamics, injecting energy and stopping the collapse.



Eventually, the dominant binary is expelled by recoil from a very violent collision. When this occurs, the cluster nucleus shrinks again until a new hard binary is formed. This gives rise to the phenomenon of *gravotermodynamic pulses*.

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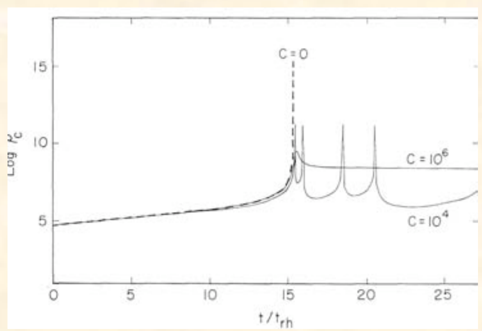

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It is ironic that the collapse of an *N*-body system (cluster) is stopped by another *N*-body system (binary) which lies, at a much smaller scale, within the first one.

The *N*-body problem in Astronomy

The British astronomer Donald Lynden-Bell, has amusingly summarized this strange behavior of binaries in clusters as follows:

Lynden-Bell Laws

1. The rate of hardening of a criminal is independent of his hardness, once he has become a hard criminal.
2. The most violent criminals commit few crimes, but these are very violent. Softer criminals commit more crimes, but these are less violent.
3. Small societies evolve until they are dominated by a very violent criminal that expels other members of the society. Eventually, this criminal is expelled too when interacting violently with another society member.
4. All societies develop a small clique that dominates them. Eventually, this group may become dominated by a very violent criminal.

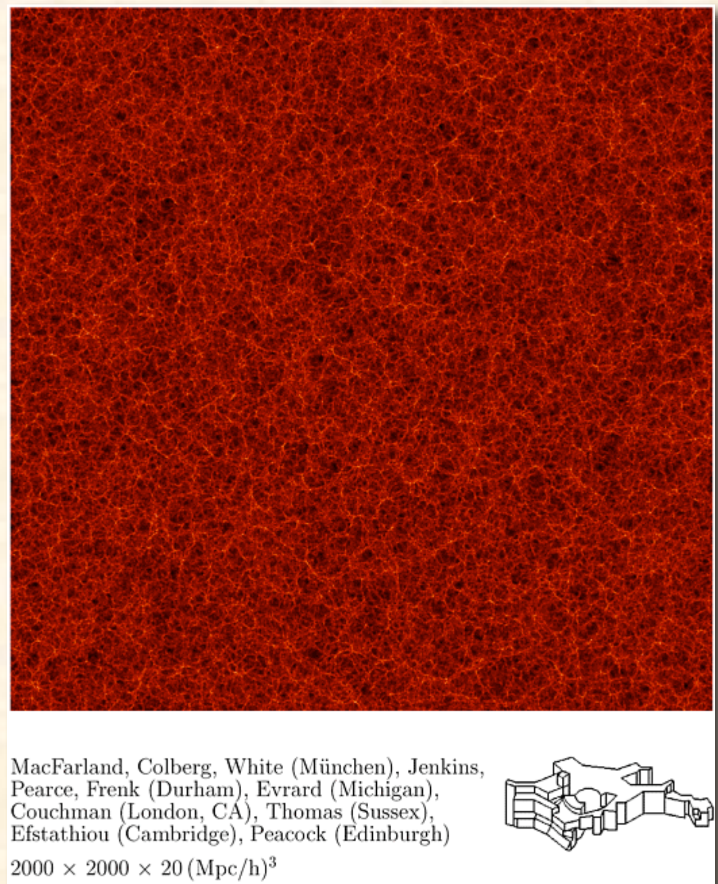
The *N*-body problem in Computer Sciences

The *N*-body problem in Computer Sciences

Some of the largest simulations run in supercomputers are *N*-body simulations

Hubble simulation: Cosmological simulation of a region of the Universe with a volume equal to 430 *Mpc*^{3}. The simulation used 10^9 particles and was run in a Cray T3E supercomputer.

The simulated volume is so vast, that light takes about 140 million years to cross it from side to side.



The *N*-body problem in Computer Sciences

The N-body problem has lead to some very ingenious solutions, as the one devised by Erik Holmberg in 1941 when he performed the first simulation of the collision of two galaxies.

THE ASTROPHYSICAL JOURNAL

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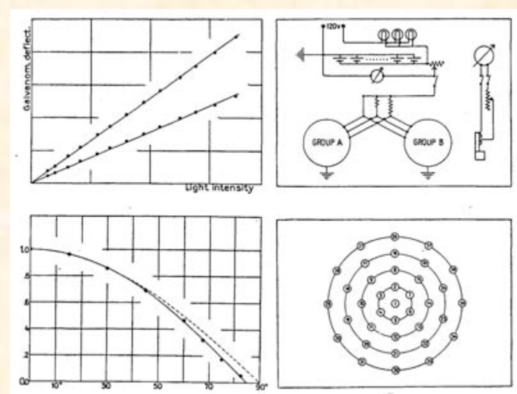
ON THE CLUSTERING TENDENCIES AMONG THE NEBULAE

II. A STUDY OF ENCOUNTERS BETWEEN LABORATORY MODELS OF STELLAR SYSTEMS BY A NEW INTEGRATION PROCEDURE

ERIK HOLMBERG

ABSTRACT

In a previous paper[1] the writer discussed the possibility of explaining the observed clustering effects among extragalactic nebulae as a result of captures. The present investigation deals with the important problem of whether the loss of energy resulting from the tidal disturbances at a close encounter between two nebulae is large enough to effect a capture. The tidal deformations of two models of stellar systems, passing each other at a small distance, are studied by reconstructing, piece by piece, the orbits described by the individual mass elements. The difficulty of integrating the total gravitational force acting upon a certain element at a certain point of time is solved by replacing gravitation by light. The mass elements are represented by light-bulbs, the candle power being proportional to mass, and the total light is measured by a photocell (Fig. 1). The nebulae are assumed to have a flattened shape, and each is represented by 37 light-bulbs. It is found that the tidal deformations cause an increase in the attraction between the two objects, the increase reaching its maximum value when the nebulae are separating, i.e., after the passage. The resulting loss of energy (Fig. 6) is comparatively large and may, in favorable cases, effect a capture. The spiral arms developing during the encounter (Figs. 4) represent an interesting by-product of the investigation. The direction of the arms depends on the direction of rotation of the nebulae with respect to the direction of their space motions.



The *N*-body problem in Computer Sciences

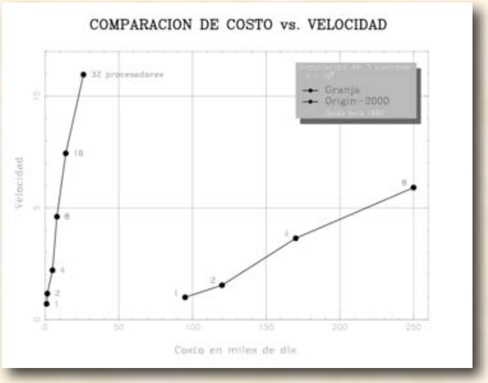
Beginning in the early 90's, a research group at the University of Tokyo, have built special purpose computers (GRAPE), where the Newtonian force has been hardwired in their circuits.

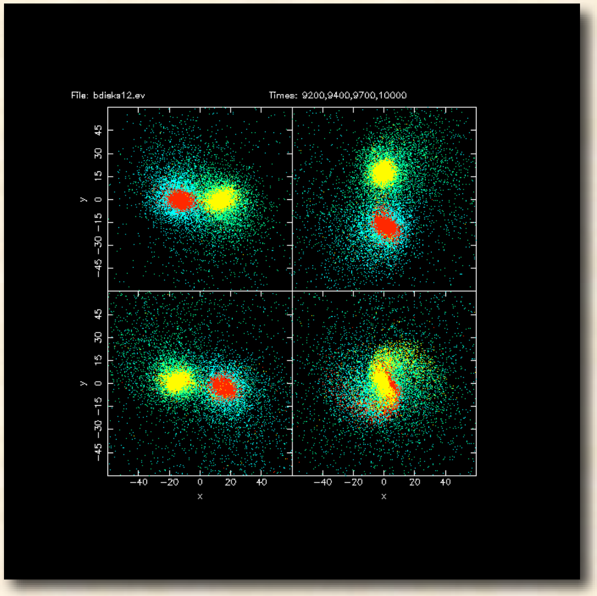
Grape-5	1999	~ 1 Tflop
Grape-6	2002	64 Tflops
Grape-DR	2008	2 Pflops?

These computers have reached very high computational speeds.

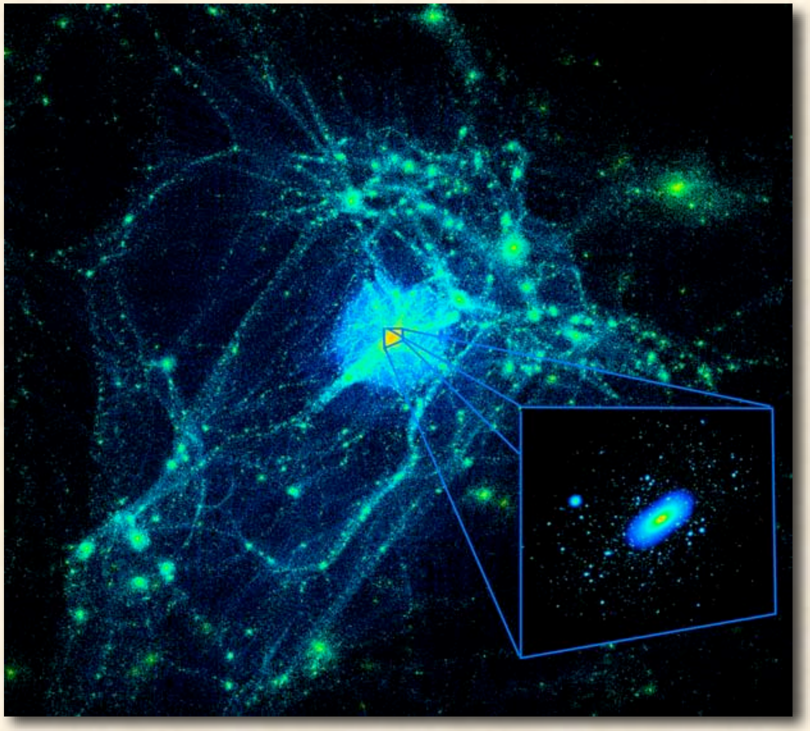
The *N*-body problem in Computer Sciences

In my home institution, we have built a 32-processor Pentium cluster that allowed us to break the one-million particle barrier.





The *N*-body problem in Cosmology



The *N*-body problem

The N-body problem is, perhaps, the oldest and most fruitful unsolved problem in the history of science.