2

Dynamics

2.1 Introduction

This chapter develops several basic ideas of dynamics, emphasizing general principles that are useful in classifying the behaviour of dynamical systems. The reader is assumed to be familiar with elementary concepts of classical mechanics. Concepts developed here will be needed in the study of special relativity (Chap. 3), statistical mechanics (Chap. 5), general relativity (Chap. 11), Sun and solar system (Vol. II), binary stars (Vol. II), and galactic dynamics (Vol. III).

2.2 Time Evolution of Dynamical Systems

Many systems encountered in nature can be described by a finite set of N real variables $[q_1(t), q_2(t), \ldots, q_i(t), \ldots, q_N(t)]$ that evolve in time. For example, in the study of two stars, moving under the influence of their mutual gravitational force, we are interested in the positions of the stars as functions of time. The position of each star can be described by three coordinates (in three-dimensional space) so that the full system can be described by a total of six functions of time. The quantities $q_i(t)$ (with $i = 1, 2, \ldots, N$) are called dynamical variables; obviously, we are free to choose any other set of N independent, single-valued functions of q_i as dynamical variables to describe the system, with the particular choice often dictated by mathematical convenience. The central problem of dynamics is related to determining the time dependence of $q_i(t)$ and studying the general characteristics of motion.

It turns out that the evolution of any physical system can be described in terms of differential equations that are (at most) second order in time in some suitably chosen variables. This implies that we could determine the functions $q_i(t)$ for all $t > t_0$ if we know the $2N$ quantities $[q_1(t_0), q_2(t_0), \ldots, q_N(t_0), \dot{q}_1(t_0), \dot{q}_2(t_0), \ldots, \dot{q}_N(t_0)]$ at $t = t_0$. In general, the initial conditions could also be specified in terms of any other set of $2N$ functions that are independent and invertible (in a unique manner) to provide the above set. Because t_0 is arbitrary, it is clear that

the state of the system at any given moment can be specified uniquely by giving $2N$ independent quantities; that is, the state of the system may be described by a point in a $2N$-dimensional space, called phase space, at any given instant. The evolution of the system is described by a curve in this $2N$-dimensional space.

Although this curve is unique, the coordinates used to describe it do not have any unique significance and are essentially decided by mathematical convenience. It is conventional to introduce in this space a general set of $2N$ coordinates denoted by $[q_1(t), q_2(t), \ldots, q_i(t), \ldots, q_N(t); p_1(t), p_2(t), \ldots, p_i(t), \ldots, p_N(t)]$ that provide the same amount of information as the dynamical variables and their first-time derivatives. Because the N-dynamical variables satisfy second-order differential equations, there must exist a set of $2N$ first-order differential equations governing the evolution of the $2N$ variables $[q_i, p_j]$. The form of these equations depends on the nature of the system and on the choice of coordinates in the phase space. We are concerned with systems called Hamiltonian systems that satisfy the following criterion: For these systems, it is possible to choose (in many different ways) the coordinates in phase space such that the differential equations can be cast in the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2.1}$$

where $H(p, q)$ is a given function called the Hamiltonian. (We will omit the indices i, etc., on q_i and merely write q for q_i, etc., when no confusion is likely to arise. Also note that we use the symbols q_i and q^i, etc., interchangeably in this chapter.) As we shall see in Section 2.4, this does *not* uniquely specify the coordinates in phase space and a wide class of them are still possible. The N quantities q_i are the dynamical variables and the second set of N quantities, p_j, are called canonical momenta.

The Hamiltonian is assumed to be an additive quantity in the following sense: The total Hamiltonian H for a system made of two noninteracting subcomponents A and B, with variables $[q^{(A)}, p^{(A)}]$ and $[q^{(B)}, p^{(B)}]$, will be taken to be the sum of the individual Hamiltonians, i.e., $H[p^{(A)}, p^{(B)}; q^{(A)}, q^{(B)}] = H^{(A)}[p^{(A)}, q^{(A)}] + H^{(B)}[p^{(B)}, q^{(B)}]$.

The time evolution of any other function, $f(p, q, t)$, defined on the phase space (and possibly depending explicitly on time) can be computed using Eq. (2.1). We have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i, \tag{2.2}$$

where the second equality is based on the summation convention, which states that any index that is repeated in a term should be summed over. By using

Eqs. (2.1) we can write this as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [H, f], \tag{2.3}$$

where the quantity $[H, f]$ – called the Poisson bracket between H and f – is defined as

$$[H, f] \equiv \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right) f. \tag{2.4}$$

This is a linear first-order differential operator acting on f. In general, we can define the Poisson bracket between any two functions g and f of phase-space coordinates by the relation

$$[g, f] = \left(\frac{\partial g}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial g}{\partial q_i} \frac{\partial}{\partial p_i} \right) f. \tag{2.5}$$

It can be easily verified that the Poisson brackets have the following properties:

$$[f, g] = -[g, f], \tag{2.6}$$

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g], \tag{2.7}$$

$$[g, f_1 f_2] = [g, f_1] f_2 + [g, f_2] f_1, \tag{2.8}$$

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0. \tag{2.9}$$

The usefulness of the Poisson brackets is closely related to the fact that the dynamical equations governing the system can be derived from a Hamiltonian. It is obvious from Eq. (2.3) that $(dH/dt) = (\partial H/\partial t) = 0$ if the Hamiltonian does not explicitly depend on time; thus the Hamiltonian is a conserved quantity for a closed system.

Exercise 2.1

Filling in the details. Prove Eqs. (2.6)–(2.9). A more transparent notation for a Poisson bracket is $D_g f \equiv [g, f]$, which shows that $[g, f]$ is the action of a differential operator (dependent on g) on f. Rewrite Eqs. (2.6)–(2.9) in this notation and interpret the results.

The dynamical equations (2.1) can also be obtained from a variational principle, which turns out to be very useful. To obtain this variational principle, let us consider the value of the quantity (called action)

$$\mathcal{A} = \int_{q_1, t_1}^{q_2, t_2} dt [p_i \dot{q}^i - H(p, q)] = \int_{q_1, t_1}^{q_2, t_2} [p_i dq^i - H(p, q)\, dt] \tag{2.10}$$

for all possible functions $p_i(t), q^i(t)$ that satisfy the condition $q^i(t_1) = q_1^i$, $q^i(t_2) = q_2^i$. The evolution of the system is postulated to be such that it makes

the value of $\mathcal{A}$ a local extremum. Setting the variation of $\mathcal{A}$ to zero, we get

$$0 = \delta\mathcal{A} = \int_{\mathcal{P}_1}^{\mathcal{P}_2} dt \left(\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial H}{\partial q_i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p^i \right)$$

$$= p_i \delta q^i \Big|_{\mathcal{P}_1}^{\mathcal{P}_2} + \int_{\mathcal{P}_1}^{\mathcal{P}_2} dt \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p^i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q^i \right], \quad (2.11)$$

where $\mathcal{P}_1 = (q_1, t_1)$, etc. The first term vanishes because δq vanishes at the end points. The second term can vanish for arbitrary δp^i and δq^i if and only if Eqs. (2.1) are satisfied, thereby proving the equivalence of the two descriptions. Note that the variational principle only demands that the end values of the dynamical variables q_i be fixed; the momenta p_j are not constrained in any manner.

For a wide class of Hamiltonians, it is possible to invert the relation $\dot{q}_i = (\partial H / \partial p_i)$ and express p_i in terms of $\dot{q}_j$. In that case the quantity $L = p\dot{q} - H$, called the Lagrangian, can be expressed as a function of q and $\dot{q}$; i.e., $L = L(q, \dot{q})$. Then the action principle involves extremizing the quantity

$$\mathcal{A} = \int_{\mathcal{P}_1}^{\mathcal{P}_2} L(q, \dot{q})\, dt \quad (2.12)$$

with respect to all functions $q(t)$ that satisfy the conditions $q(t_1) = q_1$, $q(t_2) = q_2$. The variation now gives

$$\delta\mathcal{A} = \int_{\mathcal{P}_1}^{\mathcal{P}_2} dt \left(\frac{\partial L}{\partial q_i} \delta q^i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}^i \right)$$

$$= \int_{\mathcal{P}_1}^{\mathcal{P}_2} dt \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q^i + \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q^i \Big|_{\mathcal{P}_1}^{\mathcal{P}_2}. \quad (2.13)$$

The second term vanishes because $\delta q = 0$ at the end points, giving

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right). \quad (2.14)$$

Thus, in the case of systems for which p_i can be expressed in terms of $\dot{q}_j$, we can study the dynamics either by using the Hamiltonian $H(p, q)$ or by using the Lagrangian $L(q, \dot{q})$. Given the Lagrangian $L(q, \dot{q})$, we can obtain the Hamiltonian by first computing the quantities $p = (\partial L / \partial \dot{q})$ and then computing $H = p\dot{q} - L$ and finally expressing the $\dot{q}$'s in terms of p and q. It follows from the definition that the Lagrangian is also additive in the same sense as the Hamiltonian.

Equation (2.13) also allows us to draw another important conclusion. Suppose we evaluate the value of the action $\mathcal{A}[q_2^i, t_2; q_1^i, t_1]$ along the actual trajectory of the particle and ask how this function varies when the end point q_2^i

is varied. In this case, the first term in Eq. (2.13) vanishes as the integrand is identically zero along the actual trajectory; from the second term, we find that $(\partial \mathcal{A}/\partial q_2^i) = (\partial L/\partial \dot{q}^i) = p_2^i$. Denoting the upper limit of integration by just (q, t) and treating $\mathcal{A}[q, t]$ as a function of the end point, we have the relation

$$p_i = \frac{\partial \mathcal{A}}{\partial q_i} = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.15}$$

Thus we can determine both the equations of motion and the canonical momenta from the action principle based on the Lagrangian. This result will be of use in later chapters.

The Lagrangian is not unique in the sense that it is possible to come up with very different Lagrangians that lead to the same set of differential equations (see exercises 2.3 and 2.4). One simple example of this nonuniqueness is provided by two Lagrangians that differ by a total time derivative $[dF(q, t)/dt]$ of a function $F(q, t)$ of the dynamical variables and (possibly) time. Such a total time derivative contributes to the action a quantity that is independent of the path and hence does not affect the equations of motion. But this does change the definition of canonical momenta.

Exercise 2.2

Symmetries and conservation laws: (1) Show that the quantity $E(q, \dot{q}) \equiv \dot{q}(\partial L/\partial \dot{q}) - L$ is conserved if L does not explicitly depend on time. This quantity E, called the energy of the system, is numerically the same as the Hamiltonian; note, however, that the Hamiltonian is a function of q and p whereas E is a function of q and $\dot{q}$.

(2) A Lagrangian for a free particle cannot depend on the position of the particle $\mathbf{x}$ or the direction of the velocity; so $L = L(v^2)$. Further, let us assume that the equations of motion are invariant under a Galelian transformation, defined by $\mathbf{x}_a \to \mathbf{x}_a + \mathbf{V}t$. This requires $L(v^2)$ to change (at most) by a total time derivative under Galelian transformations. Show that this is possible only if $L \propto v^2$. It is conventional to write $L = (1/2)mv^2$, where m is called the mass of the particle. Also show that for a system of interacting particles, we can maintain Galelian invariance if the Lagrangian is taken to be of the form

$$L = \sum \frac{1}{2} m_a v_a^2 - V(\mathbf{x}_1 - \mathbf{x}_2, \ldots, \mathbf{x}_a - \mathbf{x}_b, \ldots,), \tag{2.16}$$

where the function V depends on only the pairwise separation of the particle coordinates. (This result shows how symmetry considerations allow us to determine the form of the Lagrangian. We shall see several more examples of this feature in Chaps. 3 and 11.)

(3) A Lagrangian L, describing the interaction of N particles, depends on the coordinates $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_a, \ldots, \mathbf{x}_N$ and the corresponding time derivatives $\dot{\mathbf{x}}_a$, $a = 1, 2, \ldots, N$ as in part (2) above. If the origin of the coordinate system is shifted by an amount $\mathbf{l}$ so that $\mathbf{x}_a \to \mathbf{x}_a + \mathbf{l}$, then the Lagrangian should remain invariant for a closed system. Show that this implies the conservation of the quantity $\mathbf{P} = \sum_a m_a \mathbf{v}_a$, called the total momentum of the system.

(4) Consider next an infinitesimal rotation of the coordinate system by an angle $\delta\Omega$ about some axis with unit vector $\mathbf{n}$. This changes the coordinate by $\delta\mathbf{x}_a = \delta\Omega\, \mathbf{n} \times \mathbf{x}_a \equiv$

$\delta\boldsymbol{\Omega} \times \mathbf{x}_a$. For a closed system, this change should not have any effect on the dynamics. Show that this implies the conservation of the quantity $\mathbf{J} \equiv \sum m_a \mathbf{x}_a \times \mathbf{v}_a$, called the total angular momentum of the system. E, $\mathbf{P}$, and $\mathbf{J}$ are all additive quantities in the same sense as the Lagrangian.

Exercise 2.3

Equivalent Lagrangians: Consider a Lagrangian $L(q, \dot{q})$ given by

$$L = \dot{q} \int^{\dot{q}} \frac{d\alpha}{\alpha^2} F(\epsilon); \quad \epsilon \equiv \left[\frac{1}{2}\alpha^2 + V(q)\right], \tag{2.17}$$

where $F(\epsilon)$ is an arbitrary nonconstant function of its argument. Show that this Lagrangian leads to the equation of motion $\ddot{q} = -(dV/dq)$, independently of the choice for F. This result illustrates that there can be several different Lagrangians that lead to the same equation of motion.

Exercise 2.4

More on equivalent Lagrangians: Given a Lagrangian $L = L(q, \dot{q})$, we can construct a modified Lagrangian L' defined by the relation

$$L' = L - \frac{d}{dt}\left(q\frac{\partial L}{\partial \dot{q}}\right). \tag{2.18}$$

Show that varying the path $q(t)$ in L', keeping the momentum $(\partial L/\partial \dot{q})$ fixed at the end points, will lead to the same equations of motion as varying the path $q(t)$ in L keeping the coordinates fixed at end points. What is the crucial difference between L and L'?

Exercise 2.5

Maupertuis' principle: For a particle moving with constant energy $E = H(p, q)$, the relevant part of the action that should be varied, is the first term in Eq. (2.10). Show that this variation can lead to the differential equation for the spatial trajectory of the particle moving with energy E.

2.3 Examples of Dynamical Systems

As examples of simple dynamical systems, we consider some important cases that find application in different areas of astrophysics. All these Lagrangians have the form of Eq. (2.16).

2.3.1 Motion under a Central Force

As the first example, let us study a particle moving in a two-dimensional plane under the action of a central force derivable from a potential $V(r)$. The dynamical variable describing the system can be taken to be the two (polar) coordinates $q_i = (r, \theta)$. The phase space is four dimensional and has the coordinates $(r, \theta; p_r, p_\theta)$ and the Hamiltonian is given by the function

$$H(p, q) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r) \tag{2.19}$$

in suitable units. The equations of motion, corresponding to Eqs. (2.1), are

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}, \quad \dot{p}_r = \frac{p_\theta^2}{mr^3} - V'(r), \quad \dot{p}_\theta = 0. \tag{2.20}$$

Given the initial position in phase space, these equations can be integrated to determine the future time evolution of the system.

For this particular case, we can invert the relations $\dot{q}_i = (\partial H/\partial p_i)$ and obtain the Lagrangian:

$$L(q, \dot{q}) = \frac{1}{2}m\,(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{2.21}$$

The second-order differential equations, corresponding to Eq. (2.14), are

$$\frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - V'(r); \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0. \tag{2.22}$$

These equations can be integrated if we know the coordinates and their first derivatives at any given instant of time.

2.3.2 *Motion in a Rotating Frame*

As a second example, let us consider the motion of a particle acted on by a potential $V(\mathbf{x})$ but viewed from a frame of reference that is rotating with a constant angular velocity $\boldsymbol{\Omega}$. To obtain the Lagrangian for such a system, we begin with the Lagrangian in the inertial frame given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x}) \tag{2.23}$$

and note that the velocities transform under rotation by the law

$$\mathbf{v}_{\text{inertial}} = \mathbf{v}_{\text{rot}} + \boldsymbol{\Omega} \times \mathbf{x}, \tag{2.24}$$

where the subscripts inertial and rot denote the velocities in the inertial and the rotating frames, respectively. Substituting Eq. (2.24) into Eq. (2.23), we find the Lagrangian in the rotating frame to be

$$L = \frac{1}{2}m\mathbf{v}^2 + m\mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{x}) + \frac{1}{2}m(\boldsymbol{\Omega} \times \mathbf{x})^2 - V(\mathbf{x}), \tag{2.25}$$

where all quantities are measured in the rotating frame and $\mathbf{v} \equiv \mathbf{v}_{\text{rot}}$. By varying this Lagrangian, we get the equations of motion:

$$m\frac{d\mathbf{v}}{dt} = -\frac{\partial V}{\partial \mathbf{x}} + 2m\mathbf{v} \times \boldsymbol{\Omega} + m\boldsymbol{\Omega} \times (\mathbf{x} \times \boldsymbol{\Omega}). \tag{2.26}$$

The second term on the right-hand side is called Coriolis force, and the third term is the centrifugal force. (This analysis also shows that the Lagrangian or Hamiltonian description is not tied to inertial frames of reference and can be

easily generalised for any coordinate system.) The momentum and energy are now given by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + m\boldsymbol{\Omega} \times \mathbf{x}, \quad E = \frac{1}{2}mv^2 - \frac{1}{2}m\,(\boldsymbol{\Omega} \times \mathbf{x})^2 + V. \quad (2.27)$$

Note that rotation changes the form of the potential energy by adding to it a term $[-(1/2)m(\boldsymbol{\Omega} \times \mathbf{x})^2]$ that is quadratic in the coordinates.

For a system of particles, constituting a rigid body, the Lagrangian in Eq. (2.25) has to be summed over all the particles in the system (with the same $\boldsymbol{\Omega}$ for all particles). Then $\mathbf{v}_{\text{rot}} = \mathbf{V}$ gives the velocity of the centre of mass of the rigid body and $\boldsymbol{\Omega}$ denotes the angular velocity of rotation of the body. In that case, the kinetic energy will be

$$T = \sum \frac{1}{2}m\,(\mathbf{V} + \boldsymbol{\Omega} \times \mathbf{x})^2 = \sum \frac{1}{2}mV^2 + \sum m\mathbf{V} \cdot \boldsymbol{\Omega} \times \mathbf{x} + \sum \frac{1}{2}m\,(\boldsymbol{\Omega} \times \mathbf{x})^2, \quad (2.28)$$

where the summation is over all particles in the rigid body. The first term is the kinetic energy of the centre of mass motion, and the second term, which can also be expressed as $(\mathbf{V} \times \boldsymbol{\Omega}) \cdot \sum m\mathbf{x}$, vanishes when $\mathbf{x}$ is measured from the centre of mass. Therefore the rotation of the system essentially contributes only the third term to the kinetic energy. This term can be written in the form

$$T_{\text{rot}} = \frac{1}{2}I_{ik}\Omega_i\Omega_k, \quad I_{ik} = \sum m(x^2\delta_{ik} - x_i x_k). \quad (2.29)$$

The symmetric tensor I_{ik} is called moment of inertia tensor and plays a role in rotation that is similar to the one played by the mass in the case of translational motion. In the expression for T_{rot}, summation over $i, k = 1, 2, 3$ is assumed as per summation convention. The sum in the expression for I_{ik} is over all particles in the system. In the continuum limit, I_{ik} becomes

$$I_{ik} = \int d^3\mathbf{x}\,\rho(\mathbf{x})\,(x^2\delta_{ik} - x_i x_k). \quad (2.30)$$

The angular momentum of the system, measured with respect to the centre of mass of the body, is given by

$$(M)_i = \Big[\sum m\mathbf{x} \times (\boldsymbol{\Omega} \times \mathbf{x})\Big]_i = \sum m[x^2\boldsymbol{\Omega} - \mathbf{x}(\mathbf{x} \cdot \boldsymbol{\Omega})]_i = I_{ik}\Omega_k. \quad (2.31)$$

Any symmetric second-rank tensor can be reduced to the diagonal form by a suitable choice of axis. These directions for a given rigid body are called the principal axes of inertia and the corresponding values I_1, I_2, and I_3 are called the principal moments of inertia. The moment I_1, for example, will be

$$I_1 \equiv I_{11} = \sum m\big(x^2 - x_1^2\big) = \sum m\big(x_1^2 + x_3^2\big) = \sum mx_\perp^2,$$

where $\mathbf{x}_\perp$ is the perpendicular distance to mass m from the x_1 axis. This is the usual formula for computing moments of inertia of bodies with significant degree of symmetry. Clearly T_{rot} becomes the sum of three diagonal terms in this case, and we also have $M_1 = I_1 \Omega_1$, etc., in this limit.

From the variational principle applied to the basic Lagrangian, it is trivial to verify that the angular momentum of the body changes in accordance with the law $(d\mathbf{M}/dt) = \mathbf{K}$, where $\mathbf{K} = -\sum \mathbf{x} \times \nabla V$ is called the torque, corresponding to the force $\mathbf{F} = -\nabla V$ derived from the potential $V(\mathbf{x})$. Since the time derivative of any vector $\mathbf{A}$ in the inertial and the rotating frames differs by the term $\boldsymbol{\Omega} \times \mathbf{A}$, the equation of motion for the angular momentum in the rotating frame is given by

$$\frac{d'\mathbf{M}}{dt} + \boldsymbol{\Omega} \times \mathbf{M} = \mathbf{K}, \tag{2.32}$$

where the prime on (d'/dt) is a reminder of the fact that the derivative is evaluated in the moving system of coordinates. Taking the components of this equation along the principle axis of the system, we get

$$\begin{aligned} I_1 \frac{d\Omega_1}{dt} + (I_3 - I_2)\Omega_2 \Omega_3 &= K_1, \\ I_2 \frac{d\Omega_2}{dt} + (I_1 - I_3)\Omega_3 \Omega_1 &= K_2, \\ I_3 \frac{d\Omega_3}{dt} + (I_2 - I_1)\Omega_1 \Omega_2 &= K_3. \end{aligned} \tag{2.33}$$

These are called Euler's equations and govern the rotational motion of a rigid body.

As an example of the use of Eqs. (2.33), let us consider two simple cases: (1) To begin with, if the body is totally symmetric with $I_1 = I_2 = I_3$, then its free motion (corresponding to $\mathbf{K} = 0$) is trivial and is given by a uniform rotation $\boldsymbol{\Omega}$ = constant about some axis. (2) A more interesting case is the free motion of a system with axial symmetry, for which $I_1 = I_2 \neq I_3$. Then Ω_3 = constant and Ω_1 and Ω_2 satisfy the equation $\dot{\Omega}_1 = -\omega \Omega_2$, $\dot{\Omega}_2 = \omega \Omega_1$, where $\omega = \Omega_3(I_3 - I_1)/I_1$ is a constant. Multiplying one of the equations by i and adding to the second equation, we can easily solve the set and obtain $\Omega_1 = A \cos \omega t$; $\Omega_2 = A \sin \omega t$. The motion therefore consists of the vector $\boldsymbol{\Omega}$ precessing with an angular velocity ω around the symmetry axis of the body while remaining constant in magnitude. In the case in which all the principal moments of inertia are unequal, the motion can be extremely complex.

Exercise 2.6

Effects of Coriolis force: (1) Find the deflection of a freely falling body from the vertical, produced because of the rotation of the Earth, to the lowest order of approximation. (2) Determine the effect of earth's rotation on the plane of oscillation of a pendulum to

the lowest order of approximation. [Answers: (1) Taking the gravitational potential to be $U = -m\mathbf{g}\cdot\mathbf{r}$, we need to solve the equation of motion $\dot{\mathbf{v}} = 2\mathbf{v}\times\boldsymbol{\Omega} + \mathbf{g}$, where we have ignored the quadratic terms in $\boldsymbol{\Omega}$. Ignoring the term that is due to rotation, solving for $\mathbf{v}$, substituting into this equation, and solving again, we get the solution as

$$\mathbf{r} = \mathbf{h} + \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \frac{1}{3}t^3\mathbf{g}\times\boldsymbol{\Omega} + t^2\mathbf{v}_0\times\boldsymbol{\Omega}. \tag{2.34}$$

If the z axis points vertically upwards and the x axis towards the pole and λ is a northern latitude, the net deflection is given by $y = -(1/3)(2h/g)^{3/2}g\Omega\cos\lambda$, with the negative sign indicating eastward deflection. (2) Ignoring the vertical displacement of the pendulum and terms quadratic in Ω, the equations of motions are $\ddot{x} + \omega^2 x = 2\Omega_z\dot{y}$, $\ddot{y} + \omega^2 y = -2\Omega_z\dot{x}$, where Ω is the original frequency of the pendulum and $\Omega_z = \Omega\sin\lambda$. The solution to this equation is concisely given by $x(t) + iy(t) = [x_0(t) + iy_0(t)]\exp(-i\Omega_z t)$, where $x_0(t)$, $y_0(t)$ is the trajectory of the pendulum in the absence of Earth's rotation. It is clear that the net effect of rotation is to cause a shift in the plane of rotation at the rate $\Omega_z = \Omega\sin\lambda$.]

Exercise 2.7

Feynman's plate: A thin plate is thrown into the air spinning with a slight wobble. Show that it will wobble at a rate that is approximately twice as fast as it rotates. (In Feynman's memoirs, he says that "the medallion rotates twice as fast as the wobble rate"; he has mistakenly interchanged the rotation rate and the wobble rate.)

2.3.3 The Reduced Three-Body Problem

To see the effect of rotation in a physical context, we consider the motion of three-point particles under their mutual gravitational attraction subject to the following approximations: (1) We assume that, of the three bodies, the first two, with masses m_1 and m_2, are heavy and the third one, with mass m_3, is a test particle, i.e., $m_3 \ll m_1, m_2$. (2) We take m_1 and m_2 to describe circular orbits around their common centre of mass; this orbit is assumed to be unaffected by m_3. (3) The motion of m_3 takes place in the orbital plane of m_1 and m_2. This problem (called the reduced planar three-body problem) is of interest in astrophysics because it is tractable and approximates more realistic situations. For example, the motion of an asteroid (m_3) in the mutual gravitational field of the Sun (m_1) and Jupiter (m_2) can be treated along these lines.

To simplify the notation, we make a choice of units as follows. First, we take $(m_1 + m_2)$ as the unit of mass; the two masses m_1 and m_2 can now be called $(1 - \mu)$ and μ. Next, we take the unit of length to be the constant distance (diameter of the circular orbit) between m_1 and m_2. It follows that the radii of the orbits are μ and $(1 - \mu)$ for m_1 and m_2, respectively. Finally, we choose the unit of time such that $G = 1$. From these choices, it is easy to see that the angular velocity of m_1 and m_2 is also unity.

It is convenient to use a system of axes (x, y) that corotates with m_1 and m_2 with the x axis pointing towards m_2, say. In this frame m_1 and m_2 are stationary

with fixed coordinates $(-\mu, 0)$ and $(1-\mu, 0)$ respectively. If the coordinates of m_3 are (x, y) then the equations of motion obtained from Eq. (2.26) can be easily shown to be

$$\ddot{x} - 2\dot{y} = \frac{\partial \Phi}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Phi}{\partial y}, \tag{2.35}$$

where

$$\Phi(x, y) = \frac{1}{2}(x^2 + y^2) + U + \frac{1}{2}\mu(1-\mu), \tag{2.36}$$

with

$$U = \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}, \quad \rho_1 = [(x+\mu)^2 + y^2]^{1/2}, \quad \rho_2 = [(x-1+\mu)^2 + y^2]^{1/2}. \tag{2.37}$$

By multiplying the two equations of Eqs. (2.35) by $\dot{x}$ and $\dot{y}$, respectively, and adding them, we see that

$$2\Phi - \dot{x}^2 - \dot{y}^2 = C = \text{ constant}. \tag{2.38}$$

This constant C, called the Jacobi constant, is essentially the energy defined by the second equation of Eqs. (2.27). No other integral for this system is known, and hence the solutions have to be found by numerical integration. We shall discuss some characteristics of the numerical solution in Section 2.8. Right now we concentrate on the equilibrium solutions that can be obtained directly from the properties of the Lagrangian.

Given a value for the Jacobi constant C, we must satisfy the inequality $\dot{x}^2 + \dot{y}^2 = 2\Phi - C > 0$. This shows that the motion is restricted to a region bounded by the curve $\Phi = C/2$, called the Hill curve. If we rewrite Eq. (2.36) for Φ, expressing x and y in terms of ρ_1 and ρ_2, we get

$$\Phi = \frac{3}{2} + (1-\mu)\left(\frac{1}{2} + \frac{1}{\rho_1}\right)(\rho_1 - 1)^2 + \mu\left(\frac{1}{2} + \frac{1}{\rho_2}\right)(\rho_2 - 1)^2. \tag{2.39}$$

Clearly, Φ is always positive and has a minimum value $\Phi_{\min} = 3/2$ when $\rho_1 = \rho_2 = 1$. These points (at which $\rho_1 = \rho_2 = 1$) form the vertices of an equilateral triangle with the positions of masses m_1 and m_2. It follows that if the third body is kept at either of these two points with zero velocity in the rotating frame, the configuration is stationary with the particle located at the minimum of a potential. We thus find that there exists an equilibrium solution to the restricted planar three-body problem with the three particles located at the vertices of an equilateral triangle. The configuration rotates rigidly with a constant angular velocity when viewed from the inertial frame. This case corresponds to $C = 3$ for which the Hill curve degenerates into two isolated points that are usually denoted by the symbols L_4 and L_5 (see Fig. 2.1).

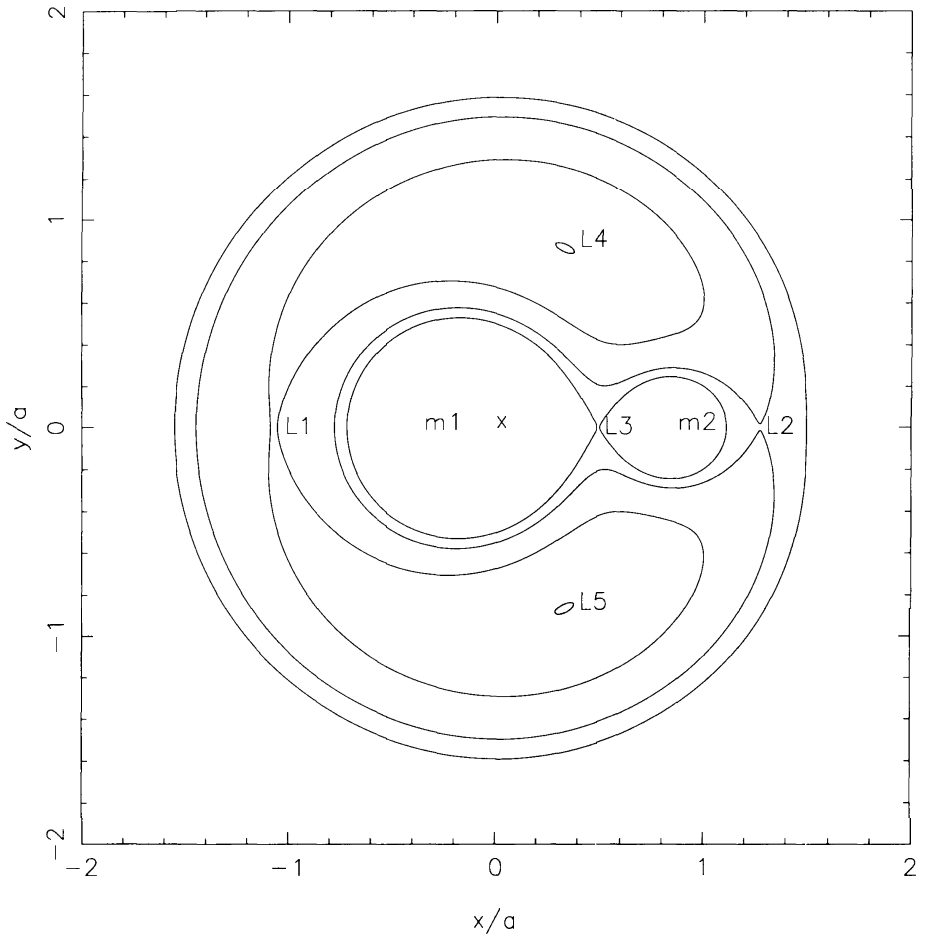

Fig. 2.1. The Lagrange points for the reduced three-body problem in a rotating frame.

More generally, the equilibrium solutions will correspond to points at which the condition $\dot{x} = 0$, $\dot{y} = 0$, $(\partial\Phi/\partial x) = 0$, and $(\partial\Phi/\partial y) = 0$ are satisfied simultaneously. These equations can be solved in a straightforward manner, and we will find that, in addition to L_4 and L_5, there are also three other points L_1, L_2, and L_3 along the line joining m_1 and m_2 at which these conditions are satisfied. It is conventional to take L_3 to be between the two masses. All these points correspond to equilibrium situations and are called Lagrange points.

The Hill curve can be thought of as an equipotential line, and its characteristics depend critically on the value of C. Figure 2.1 shows the variation of Hill curves as the value of C is increased from $C = 3$. The point marked X is the centre of mass of the system with masses m_1 and m_2 where $(m_1/m_2) = 5$. Close to the masses, the gravitational potential is dominated by the individual masses and has circular contours. The same result holds far away from the two masses. At intermediate scales, the contours for potentials have the shapes shown in the figure.

Exercise 2.8

Filling in the details. Prove Eqs. (2.35)–(2.39).

It is obvious that the Lagrange points L_1, L_2, and L_3 correspond to unstable equilibria and are local maxima of the potentials. For future reference, we give

a fitting function for the distance to the inner Lagrangian point L_3 from the two masses:

$$l_1 = a\left[0.500 - 0.227 \log_{10}\left(\frac{m_2}{m_1}\right)\right], \quad l_2 = a\left[0.500 + 0.227 \log_{10}\left(\frac{m_2}{m_1}\right)\right], \tag{2.40}$$

where a is the distance between the two masses m_1 and m_2. The stability of the equilibrium solution corresponding to the Lagrange points L_4 or L_5 is more difficult to determine. Linear stability analysis can be performed by examination of the eigenvalues of the matrix of second derivatives $(\partial^2\Phi/\partial x_i \partial x_j)$. This analysis is straightforward, although tedious, and leads to the conclusion that the solutions are not linearly unstable for $\mu(1-\mu) < (1/27)$. This corresponds to the condition

$$\mu < \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right) \approx 0.0385, \tag{2.41}$$

that is, the upper limit on the ratio of masses is $\sim 1/25$. A more complex analysis shows that the motion in the equilateral solution is actually stable if this condition is satisfied except for two specific values of the ratio μ. Thus, for all practical purposes, we can take the above criteria as the one for stability of the equilateral solution.

Exercise 2.9

Stabilising influence of rotation: A system of masses produces a gravitational potential $\phi(\mathbf{x}, t)$. Prove that such a gravitational potential cannot have a local maxima or minima at any $\mathbf{x}$ unoccupied by the mass points. Explain how the rotation, combined with gravity, can lead to a Lagrange point that is stable against small perturbations.

2.4 Canonical Transformations

It is clear from Eqs. (2.1) that the problem of Hamiltonian dynamics boils down to solving $2N$-coupled first-order ordinary differential equations. The structure of these equations will depend very much on the choice of the dynamical variables and conjugate momenta. As it may be easier to solve these equations with one particular choice rather than with another, it is important to ask whether any general formalism can be developed to facilitate solving these equations. We now discuss some of these general techniques.

It was mentioned in Section (2.2) that the form of the Eqs. (2.1) presupposes a judicious choice of coordinates in phase space. An arbitrary transformation of coordinates from the set (q, p) to some other set (q', p') will not maintain the form of Eqs. (2.1). We begin by asking the following question: What is the most general class of transformation from a set (q, p, H) to another set (Q, P, H'), that will preserve the form of equations of motion (2.1)?

Because the same dynamical equations are obtained by varying the two different actions, made from $(p\dot{q} - H)$ and $(P\dot{Q} - H')$, the two Lagrangians can differ by a total time derivative (dF/dt) of some function F, so that

$$p\dot{q} - H - (P\dot{Q} - H') = \frac{dF}{dt} \tag{2.42}$$

or

$$dF = pdq - PdQ + (H' - H)\,dt. \tag{2.43}$$

If $F = F(q, Q, t)$, it follows that

$$p_i = \frac{\partial F}{\partial q_i}, \quad P_i = -\frac{\partial F}{\partial Q_i}, \quad H' = H + \frac{\partial F}{\partial t}. \tag{2.44}$$

The function F is called the generating function for the canonical transformation from (q, p) to (Q, P). Given any function $F(q, Q, t)$, we can generate a transformation from one set of coordinates in phase space to another such that the structure of Eqs. (2.1) remains the same. In particular, if F has no explicit dependence on time, then the Hamiltonian does not change in its numerical value.

In the above case, we have taken the generating function to depend on old and new dynamical variables. It is also possible to have a generating function that depends on the old coordinates q and new canonical momenta P. To obtain the formulas for this case, we only have to note that Eq. (2.43) can be rewritten as

$$d(F + P_i Q_i) = p_i dq_i + Q_i dP_i + (H' - H)\,dt. \tag{2.45}$$

The argument of the differential on the left-hand side, expressed in terms of the variables q and P, is a new generating function $\Phi(q, P, t)$, say. Then,

$$p_i = \frac{\partial \Phi}{\partial q_i}, \quad Q_i = \frac{\partial \Phi}{\partial P_i}, \quad H' = H + \frac{\partial \Phi}{\partial t}. \tag{2.46}$$

We can also obtain similar formulas with other combinations of variables.

The existence of canonical transformations, under which the form of the dynamical equations remains invariant, denies any special status to the dynamical variables and conjugate momenta. For example, by using a generating function $F = qQ$, we can produce the transformation $Q = p$, $P = -q$, thereby interchanging the roles of coordinates and momenta.

The Poisson brackets remain invariant under the canonical transformation. Let $[f, g]_{p,q}$ be the Poisson bracket for two quantities f and g, in which the differentiation is with respect to the variables p and q, and let $[f, g]_{P,Q}$ be the bracket in which the differentiation is with respect to P and Q. Then

$$[f, g]_{p,q} = [f, g]_{P,Q}. \tag{2.47}$$

To prove this, we may argue as follows: Because time appears only as a parameter in the canonical transformation, it is sufficient to prove the invariance for quantities that do not depend explicitly on time. Let us now regard g as a

Hamiltonian for some fictitious system. Then we know that $df/dt = -[f, g]_{p,q}$. But the derivative df/dt can depend on only the properties of motion of the system and not on the choice of coordinates in phase space. Therefore the Poisson bracket $[f, g]$ is invariant under canonical transformations.

Exercise 2.10

Invariant volumes in phase space: The canonical transformation also leaves volumes of certain regions in phase space invariant. Let $d\mathcal{V} = dq_1 \cdots dq_N dp_1 \cdots dp_N$ denote an element of volume in phase space. The integral of $d\mathcal{V}$ over some region of phase space represents the volume of that region. If we now replace the variables p, q with P, Q through a canonical transformation, show that the volumes of corresponding regions of space remain equal:

$$\int \cdots \int dq_1 \cdots dq_N \, dp_1 \cdots dp_N = \int \cdots dQ_1 \cdots dQ_N \, dP_1 \cdots dP_N. \quad (2.48)$$

Next prove the invariance of a larger class of phase-space integrals of the form

$$\iint \sum_i dq_i \, dp_i, \quad \iiiint \sum_{i \neq k} dq_i \, dp_i \, dq_k \, dp_k, \ldots, \quad (2.49)$$

in which the integration is over manifolds of two, four, etc., dimensions in phase space. (These are called Poincaré invariants. Because the dynamical variables and conjugate momenta get mapped to the corresponding quantities during the time evolution, it follows that the evolution of the system itself can be thought of as a canonical transformation from the initial set of variables to the final set of variables. Because the phase volume remains invariant under any canonical transformation, it follows that the volume of the phase space remains invariant during the evolution.)

Having determined the most general set of transformation that leaves the dynamical equations invariant, we now turn to the question of using this information to solve for the dynamical evolution of the system. The simplest possible Hamiltonian we can think of is one that vanishes identically in the new set of variables, that is, $H' = 0$. In this case P and Q will be constant and the evolution will be frozen in the new coordinate system. The vanishing of the new Hamiltonian can be achieved by use of a generating function S that satisfies

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0. \quad (2.50)$$

We obtained this by setting $H' = 0$ in the third equation of Eqs. (2.46) and substituting for p in $H(p, q)$ by $p = (\partial\Phi/\partial q)$; it is conventional to use the symbol S rather than Φ for the generating function that makes $H' = 0$. This equation (called the Hamilton–Jacobi equation) determines the generating function; by using this function we can go to a new set of coordinates and vanishing Hamiltonian. Suppose that this equation can be solved to obtain a particular solution $S = S(t, q^i, \alpha_i)$, where α_i (with $i = 1, 2, \ldots, N$) are constants of

integration. This generating function, of course, makes $H' = 0$ by Eqs. (2.44) and (2.50). We now treat α_i as the new momenta P_i (which are constant in time because $H' = 0$). From Eqs. (2.46) we see that the new coordinates are $Q_i = (\partial S/\partial \alpha_i)$. But because $H' = 0$, these coordinates should also be constant; $Q^i =$ constant $\equiv \beta^i$, say. We thus get the equations $\beta_i = [\partial S(t, q_i, \alpha_i)/\partial \alpha_i]$, which implicitly determine the trajectories $q_i(t)$ in terms of the $2N$ constants (α_i, β_i). Thus we have reduced the problem of dynamics to one of solving the partial differential equation (2.50).

The generating function S has a simple physical meaning; it actually corresponds to the action $\mathcal{A}$ treated as a function of the coordinates and time at the end point, that is, the functional dependence of S on q and t will be the same as that of the quantity

$$\mathcal{A}(q, t) = \int_{q_1, t_1}^{q, t} L \, dt, \tag{2.51}$$

where the integral is taken along the extremal path that is a solution to the equation of motion. To prove this, we only have to show that

$$p = \left(\frac{\partial \mathcal{A}}{\partial q}\right), \quad H = -\left(\frac{\partial \mathcal{A}}{\partial t}\right). \tag{2.52}$$

The first relation was proved earlier; see Eq. (2.15) in Section 2.2. To prove the second relation, we note that $d\mathcal{A}/dt = L$ by definition, and we can also write

$$\frac{d\mathcal{A}}{dt} = \frac{\partial \mathcal{A}}{\partial t} + \frac{\partial \mathcal{A}}{\partial q_i}\dot{q}_i = \frac{\partial \mathcal{A}}{\partial t} + p_i \dot{q}_i. \tag{2.53}$$

Comparison gives $(\partial \mathcal{A}/\partial t) = L - p_i \dot{q}_i$ or

$$\frac{\partial \mathcal{A}}{\partial t} = -H. \tag{2.54}$$

This shows that the solution to Eq. (2.50) is actually the action treated as a function of the end points.

As an example of the use of Eq. (2.50), let us consider the motion of a particle in one dimension under the action of a potential $V(x)$. The Hamiltonian for the system is

$$H(p, x) = \frac{p^2}{2m} + V(x), \tag{2.55}$$

and the Hamilton–Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) = 0. \tag{2.56}$$

This partial differential equation can be solved by the method of separation of variables with the ansatz $S = -Et + A(x)$, where E is a constant. By substituting

this ansatz into Eq. (2.56) and rearranging terms, we can express $A(x)$ as an integral, giving

$$S = -Et + \int dx \sqrt{2m[E - V(x)]}. \tag{2.57}$$

This gives the solution to the Hamilton–Jacobi equation in terms of the coordinate x, time t, and a constant E. We think of this function as a generating function, with the new canonical momentum identified with E. We also know that, by construction, the new Hamiltonian vanishes. From our canonical transformations in Eqs. (2.46), the new coordinates are given by $(\partial S/\partial E)$. However, because the new Hamiltonian vanishes, the new coordinates are also constant in time. Equating $(\partial S/\partial E)$ to some constant t_0, we get

$$t + t_0 = \int dx \left\{ \frac{m}{2[E - V(x)]} \right\}^{1/2}. \tag{2.58}$$

This equation provides the complete solution to our problem. Given the form of $V(x)$, we can now express x as a function of t and the two constants t_0 and E. The constants can be determined if x and $\dot{x}$ are given at some initial instant of time. We thus note that the problem of a particle, moving in one dimension under the action of an arbitrary potential, can be reduced to a quadrature.

Exercise 2.11

Potentials for which period is independent of amplitude: (1) Consider a particle of mass m moving in a potential $V(x) \propto |x|^n$, where $n > 0$. The energy of the particle is such that it oscillates between $x = a$ and $x = -a$. How does the period of oscillation scale with the amplitude a? (2) In particular, note that the period is independent of amplitude for such a power-law potential only if $n = 2$. There exists, however, very many different potentials $V(x)$ (which are not pure power laws) such that a particle can oscillate about a minimum of the potential with a period independent of amplitude. Give a general characterisation of such a potential. (3) One particular example of such a potential is $V(x) \propto (x^{-2} + x^2)$. Argue why the period of oscillation should be independent of amplitude in such a potential without doing any calculation. (Hint: Think of a two-dimensional harmonic oscillator.)

As a second example, consider the motion of a particle in a plane under the action of a central force with potential $V(r)$. By use of the Hamiltonian in Eq. (2.19), the Hamilton–Jacobi equation becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right] + V(r) = 0. \tag{2.59}$$

We now substitute the ansatz $S = -Et + J\theta + A(r)$ into the Hamilton–Jacobi equation and obtain

$$A = \int dr \left[2m\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}, \tag{2.60}$$

so that

$$S = -Et + J\theta + \int dr\left[2m\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}. \tag{2.61}$$

Once again $S(t; r, \theta; E, J)$ can be thought of as a generating function for a canonical transformation depending on the old coordinates r, θ and new momenta E, J. Because the new Hamiltonian vanishes, the new coordinates, given by the derivatives of S with respect to E and J, must be constants. With these constants denoted by t_0 and θ_0, the equations $t_0 = (\partial S/\partial E)$ and $\theta_0 = (\partial S/\partial J)$ are reduced to

$$t + t_0 = \int dr \left(\frac{m}{E - V - J^2/2mr^2}\right)^{1/2}, \tag{2.62}$$

$$\theta - \theta_0 = \int dr \frac{J/r^2}{[2m(E - V - J^2/2mr^2)]^{1/2}}. \tag{2.63}$$

These equations provide the complete solution to the problem. Equation (2.62) relates r to t, and Eq. (2.63) relates θ and r. There are also four arbitrary constants, J, E, t_0, and θ_0, that need to be determined in terms of $r, \theta, \dot{r}, \dot{\theta}$ at an initial instant. The problem has again been reduced to quadrature.

The two examples above show how the method of the Hamilton–Jacobi equation works in practice. It must, however, be remembered that the success was due to the fact that we could separate the variables in the partial differential equation. It is this feature that allowed us to reduce the partial differential equation to an ordinary differential equation (for the function A) and to solve the problem. In the second example, the choice of coordinates also played a key role. If we had used the Cartesian coordinates x, y rather than the polar coordinates r, θ, the problem would have been less tractable. There has been extensive study of the separability of the Hamilton–Jacobi equation in different coordinate systems. It can be shown that there are 11 real coordinate systems that satisfy the criteria for separability, but only a few of them are useful in studying interesting physical phenomena. If the equation is not separable, then we must resort to numerical techniques, and it is better to start with Eqs. (2.1) rather than with the Hamilton–Jacobi equation.

Exercise 2.12

Separability of Hamilton–Jacobi equation in an unfamiliar coordinate system: Consider the motion of a particle under the action of a potential $U = (\alpha/r) - Fz$, which is a combination of a Coulomb field and a uniform field. (1) To study the motion of such a particle, it is convenient to use *parabolic* coordinates (ξ, ϕ, η) that are related to the standard cylindrical coordinates (ρ, ϕ, z) by the equations $z = (\xi - \eta)/2$ and $\rho = \sqrt{\xi\eta}$. Show that the Hamiltonian for a particle can be written in these coordinates as

$$H = \frac{2}{m}\frac{\xi p_\xi^2 + \eta p_\eta^2}{\xi + \eta} + \frac{p_\phi^2}{2m\xi\eta} + U(\xi, \eta, \phi). \tag{2.64}$$

(2) Show that the Hamilton–Jacobi equation is separable in these coordinates for any potential of the form

$$U = \frac{a(\xi) + b(\eta)}{\xi + \eta} = \frac{a(r+z) + b(r-z)}{2r}. \tag{2.65}$$

The potential $U = (\alpha/r) - Fz$ has this form, with $a(\xi) = \alpha - (F\xi^2/2)$, $b(\eta) = \alpha + (F\eta^2/2)$. Hence obtain the complete solution to the problem.

The fact that there are only a limited number of cases in which a complete solution of the problem is possible gives rise to two important questions that are central to the investigations in modern mechanics. Under what circumstances can we reduce a given dynamical problem to quadrature? Can we say anything about the general characteristics of motion for systems that can be reduced to quadrature? We now turn to the discussion of these questions.

2.5 Integrable Systems

To understand the nature of systems for which the problem of motion can be reduced to quadrature, we need to introduce the concept of an integral of motion. Note that, if the equations of motion can be integrated, then we can obtain the functions $q_i = q_i(t; q_i^0, p_i^0)$ and $p_i = p_i(t; q_i^0, p_i^0)$ in terms of the initial conditions (q_i^0, p_i^0). In principle we can invert these relations and express the $2N$ quantities q_i^0 and p_i^0 in terms of q_j, p_j and t. By their very construction, these $2N$ functions of q_j, p_j and t are constants, that is, their values remain the same as the system evolves. Out of these $2N$ constants, one constant is trivial as the origin of time is immaterial for a closed system. We can always change t to $(t + t_0)$ in a parameterised curve $C(t)$, keeping the physics invariant. Thus any dynamical system has $(2N - 1)$ nontrivial constants of motion that restrict the system to a $[2N - (2N - 1)] = 1$ dimensional surface in phase space. This one-dimensional surface is the trajectory $C(t)$.

Most of the $(2N - 1)$ constants of motion described above are, however, useless in the sense that these constants can be determined only after the equations of motion are integrated (and hence are of no help in studying the dynamics). For certain physical systems there might exist a subset of k constants of motion $f_A(q, p)$, $A = 1, 2, \ldots, k$ that are independent of t and that can be detected from general symmetry considerations. Such constants of motion, which are independent of time, are called integrals of motion. Even among the integrals of motion, all are not of equal importance. Certain integrals of motion isolate the time evolution of the system to a restricted region in phase space, thereby simplifying the physics. Such integrals are called isolating integrals. The integrals of motion that do not isolate the motion to a well-defined region in phase space are called nonisolating.

To illustrate these concepts, let us consider the example of a two-dimensional harmonic oscillator with the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2). \tag{2.66}$$

In this particular case, we can easily integrate the equations of motion and obtain the solutions

$$x(t) = A\cos(\omega_x t + \epsilon_x), \quad y(t) = B\cos(\omega_y t + \epsilon_y), \tag{2.67}$$

where $(A, B, \epsilon_x, \epsilon_y)$ are the four constants that are to be determined in terms of $[x(0), y(0), p_x(0), p_y(0)]$. From the above solution we get

$$\begin{aligned} x &= A\cos(\omega_x t + \epsilon_x), \quad & p_x &= -A\omega_x \sin(\omega_x t + \epsilon_x), \\ y &= B\cos(\omega_y t + \epsilon_y), \quad & p_y &= -B\omega_y \sin(\omega_y t + \epsilon_y). \end{aligned} \tag{2.68}$$

We can now express $(A, B, \epsilon_x, \epsilon_y)$ in terms of (x, p_x, y, p_y, t):

$$A^2 = x^2 + \frac{p_x^2}{\omega_x^2}, \quad B^2 = y^2 + \frac{p_y^2}{\omega_y^2}, \tag{2.69}$$

$$\epsilon_x = -\omega_x t - \tan^{-1}\left(\frac{p_x}{x\omega_x}\right), \quad \epsilon_y = -\omega_y t - \tan^{-1}\left(\frac{p_y}{y\omega_y}\right). \tag{2.70}$$

These are all constants of motion, among which ϵ_x and ϵ_y depend on t explicitly; by using the arbitrariness in the origin of t, we can eliminate one of them.

Because the Lagrangian depends on both x and y we expect only the energy to be a conserved quantity. This will define a three-dimensional region of four-dimensional phase space in which the motion takes place. The above analysis shows, however, that it is possible to write two independent integrals A and B for the system because the x and y motions are uncoupled. We have the integrals (A, B) or, equivalently, (E_x, E_y), where

$$E_x \equiv \frac{1}{2}\omega_x^2 A^2 = \frac{p_x^2}{2} + \frac{1}{2}\omega_x^2 x^2, \quad E_y \equiv \frac{1}{2}\omega_y^2 B^2 = \frac{p_y^2}{2} + \frac{1}{2}\omega_y^2 y^2. \tag{2.71}$$

[The total energy E is just $(E_x + E_y)$ and hence is not an independent integral.] These integrals clearly isolate the motion in the $x - p_x$ and $y - p_y$ planes. Because there are two integrals, the motion is confined to a $4 - 2 = 2$ dimensional surface in the phase space. The equations of this surface, given by Eqs. (2.71), can be written parametrically as

$$p_x = \sqrt{2E_x}\sin\alpha, \quad x = \sqrt{\frac{2E_x}{\omega_x^2}}\cos\alpha \quad (0 \le \alpha < 2\pi), \tag{2.72}$$

$$p_y = \sqrt{2E_y}\sin\beta, \quad y = \sqrt{\frac{2E_y}{\omega_y^2}}\cos\beta \quad (0 \le \beta < 2\pi). \tag{2.73}$$

This is the two-dimensional surface of a torus on which the motion takes place.

We can ask whether the existence of two more integrals ϵ_x and ϵ_y puts any further restriction on the nature of the motion. From solutions (2.67), it is clear that

$$\frac{\epsilon_x}{\omega_x} - \frac{1}{\omega_x}\cos^{-1}\left(\frac{x}{A}\right) = \frac{\epsilon_y}{\omega_y} - \frac{1}{\omega_y}\cos^{-1}\left(\frac{y}{B}\right) \tag{2.74}$$

or

$$\cos^{-1}\left(\frac{x}{A}\right) - \frac{\omega_x}{\omega_y}\cos^{-1}\left(\frac{y}{B}\right) = \text{constant} \equiv c. \tag{2.75}$$

This quantity c is clearly another integral of motion. But – in general – this does not isolate the region where the motion takes place any further, because $\cos^{-1} z$ is a multiple-valued function. To see this more clearly, let us write Eq. (2.75) as

$$x = A\cos\left\{c + \frac{\omega_x}{\omega_y}\left[\cos^{-1}\left(\frac{y}{B}\right) + 2\pi n\right]\right\}, \tag{2.76}$$

where $\mathrm{Cos}^{-1} z$ (with an uppercase C) denotes the principle value. For a given value of y we will get an infinite number of x's as we take $n = 0, \pm 1, \pm 2 \ldots$. Thus, in general, the curve in Eq. (2.76) will fill a region in the (x, y) plane.

A special situation arises if (ω_x/ω_y) is a rational number. In that case, the curve closes on itself after a finite number of cycles. Then c is also an isolating integral and we have three isolating integrals: (E_x, E_y, c). The motion is confined to a closed (one-dimensional) curve on the surface of the torus.

We began with a system with two dynamical variables x and y so that the phase space is four dimensional. In general, such a system will have only energy as a conserved quantity; thus, in general, we would have expected the motion to take place in a three-dimensional subspace of the full four-dimensional phase space. However, in this particular case, we had two integrals of motion, E_x and E_y. This restricts the motion to a $(4 - 2) = 2$ dimensional subspace of the phase space, which happens to be a torus. For generic values of ω_x and ω_y, the motion fills the surface of the torus. But if the ratio of the frequencies is rational, then the motion is restricted to a space of one lower dimension: in this case, to a $(2 - 1) = 1$ dimensional curve.

All these features happen to be quite generic for a large class of systems called integrable Hamiltonian systems. A system with N-dynamical variables is called integrable if there exist N independent, isolating integrals of motion with mutual Poisson brackets vanishing. (In the above example, $N = 2$ and E_x and E_y are the two isolating independent integrals of motion.) This immediately implies that the motion is confined to a $(2N - N) = N$ dimensional subspace of the phase space. More importantly, this subspace always happens to be an N-dimensional torus. Given the N integrals of motion, it is also possible to reduce the problem to quadrature by a suitable canonical transformation. We now explore these ideas in detail.

Let us consider a system with a $2N$-dimensional phase space and N independent integrals of motion, $F_m(q, p)$, where $m = 1, 2, \ldots, N$. We assume that the Poisson brackets $[F_m, F_n]$ vanish for all values of n and m. The existence of these integrals suggests the following approach to the dynamics of such a system. Given the N integrals of motion, $F_i(q, p)$ with $i = 1, 2, \ldots, N$, it is convenient to make a canonical transformation to a new set of variables (Q, P), with $P_j = F_j(q, p)$. Because F's are constants of motion, we must have

$$\dot{P}_j = -\frac{\partial \mathcal{H}}{\partial Q_j} = \dot{F}_j = 0. \tag{2.77}$$

Therefore the new Hamiltonian $\mathcal{H}$ cannot depend on Q_i and is purely a function of the P's. Let $(\partial \mathcal{H}/\partial P_j) = \Omega_j(P)$. Because the P's are independent of t, the Ω's are also independent of t, allowing the equations of motion $\dot{q}_j = (\partial \mathcal{H}/\partial P_j) = \Omega_j$ to be integrated to give

$$Q_j = \Omega_j t + \delta_j. \tag{2.78}$$

To solve the problem completely, we only have to express the old coordinates q in terms of the new coordinates Q. We do this by demanding that our transformation should be canonical with a suitable generating function that can be obtained as follows. The constancy of F_j can be expressed by the set of equations $F_j(q, p) = f_j$, where the f_j's are N constants. Inverting this equation, we can obtain p_i in terms of q's and f's as $p_i = p_i(q_j, f_k)$. The generating function can be taken to be

$$S(q, P) = \int^q p_k(q, f)\, dq^k = S(q, f), \tag{2.79}$$

with a summation over k assumed. The new coordinates Q_i are given by

$$Q_i = \frac{\partial S(q, f)}{\partial f_i}, \tag{2.80}$$

which relates the new and the old coordinates and contains N arbitrary constants. Equation (2.78) gives the new coordinates as a function of time and other N constants. Combining the two, we can find the trajectories $q_i(t)$ in terms of $2N$ arbitrary constants of motion, which completely solves the problem.

Thus, for integrable systems, the procedure for solving the dynamical problem can be summarised as follows: (1) Invert the relation $F_j(q, p) = f_j$ to obtain $p_i = p_i(q_j, f_k)$; here F_j's are the N given integrals of motion. (2) Integrate $p_i(q_j, f_k)$ over dq_i to determine the function $S(q, f)$ [see Eq. (2.79)]. (3) Use Eq. (2.80) to relate the new coordinates Q_j to the old coordinates q_i and N constants f_i. (4) The new coordinates are linear functions of time [see Eq. (2.78)] that allow us to determine the time evolution of old coordinates $q_i(t)$.

The existence of N integrals restricts the motion to an N-dimensional manifold $\mathcal{M}$. To understand the topology of this manifold $\mathcal{M}$, let us construct the following

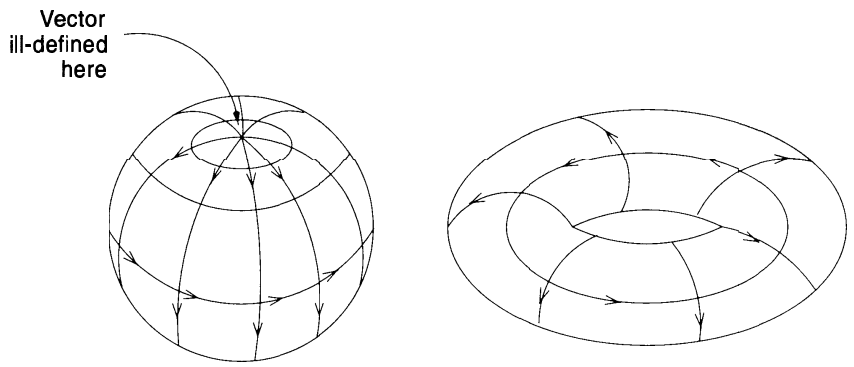

Fig. 2.2. Two parallelisable vector fields can exist on a two torus but not on a two sphere.

N vector fields V_a:

$$V_a \equiv \left(\frac{\partial F_a}{\partial p^i}, -\frac{\partial F_a}{\partial q^j} \right). \tag{2.81}$$

Note that a runs from 1 to N, specifying which vector field; for each value of a, the vector field has $2N$ components given by Eq. (2.81) so that we now have N vectors (each having $2N$ components) defined at every location in the phase space. In the submanifold $\mathcal{M}$ defined by the equation $F_i(q, p) = f_i$, the vector fields are smooth and independent; moreover, the V_a's are tangential to $\mathcal{M}$ in the sense that V_a is perpendicular to the normal of $\mathcal{M}$:

$$V_a \cdot n^b = \left(\frac{\partial F_a}{\partial p^i}, -\frac{\partial F_a}{\partial q^j} \right) \cdot \left(\frac{\partial F_b}{\partial q^i}, \frac{\partial F_b}{\partial p^j} \right) = [F_a, F_b] = 0 \tag{2.82}$$

for all a, b. If we restrict ourselves to bounded motion in which the region of phase space accessible to the particle is finite, the manifold $\mathcal{M}$ will be compact. There is a theorem in topology that states that a compact manifold that is parallelisable with N smooth independent vector fields must be an N torus. Figure 2.2 illustrates the situation for $N = 2$ and shows why the manifold could not, for example, be a sphere.

These tori are called invariant tori because an orbit starting out in one remains in it forever. A possible coordinate system on the phase space is now given by the set (Q, P), with $P = f$ labelling the torus and Q being the coordinates on the torus. This choice, however, is not yet unique because we could have started with any other function of the original integrals of the motion. There is, however, a very natural coordinatisation of the torus $\mathcal{M}$ called the action-angle variables (I, Θ). The definition of these coordinates is based on the observation that the generating function in Eq. (2.79) is generally multivalued because the momenta can be multivalued. In other words, the generating function can depend on the path chosen to define the integral in Eq. (2.79). But if two different paths in $\mathcal{M}$ are continuously deformable into one another, then the generating function must be the same for these two because it satisfies the (local) Hamilton–Jacobi equation. We may say that S is locally single valued. It immediately follows that

for all closed paths on $\mathcal{M}$ that can be shrunk to a point, $S=0$. On an N torus, there exist N independent irreducible closed paths C_a that cannot be deformed to a point. The action variables I_a are defined by

$$I_a(f)=\frac{1}{2\pi}\int_{C_a} p_i(q, f)\,dq^i, \tag{2.83}$$

where the integral is taken over these irreducible closed curves. This defines the N coordinates I_a in terms of the f's and vice versa. Because the Hamiltonian is one of the f's, we can invert this relation and express the Hamiltonian as a function of the I_a's: $H=H(I)$.

Once we have the actions I_a defining the torus, the coordinates on the torus are defined to be the angles $\Theta_a=[\partial S(q, I)/\partial I_a]$. [Note that we are now expressing the action S in terms of (q, I) by expressing the f's in terms of I.] It can be easily verified that as we traverse the closed path C_a, the angle Θ_a changes by 2π while all other angles Θ_b with $b\neq a$ remain the same. Because of this feature, the canonical transformation from q, p to Θ, I expresses the original coordinates as periodic functions of Θ with period 2π:

$$q=\sum_{\mathbf{m}} q_{\mathbf{m}}(I)\exp(i\mathbf{m}\cdot\Theta),\qquad p=\sum_{\mathbf{m}} p_{\mathbf{m}}(I)\exp(i\mathbf{m}\cdot\Theta). \tag{2.84}$$

To illustrate the above concepts, let us work out two examples by using action-angle variables. We start with the case of the two-dimensional harmonic oscillator with the Hamiltonian

$$H(q, p)=\frac{p_1^2}{2}+\frac{p_2^2}{2}+\frac{\omega_1^2 q_1^2}{2}+\frac{\omega_2^2 q_2^2}{2}. \tag{2.85}$$

The two constants of motion are given by

$$F_1=\frac{p_1^2}{2}+\frac{\omega_1^2 q_1^2}{2},\qquad F_2=\frac{p_2^2}{2}+\frac{\omega_2^2 q_2^2}{2}. \tag{2.86}$$

To effect a canonical transformation from (q, p) to (Θ, I), we have to find the actions defined by Eq. (2.83). They are

$$I_a=\frac{1}{2\pi}\oint p_i dq^i=\frac{1}{2\pi}\oint\sqrt{2\left(F_a-\frac{1}{2}\omega_a^2 q_a^2\right)}\,dq_a=\frac{F_a}{\omega_a}\quad (a=1, 2). \tag{2.87}$$

Because the original Hamiltonian is just $H=F_1+F_2$, we find that the Hamiltonian can be expressed in terms of the actions as

$$H(I)=F_1+F_2=I_1\omega_1+I_2\omega_2. \tag{2.88}$$

From the general theory of action angle-variables, we know that

$$\dot{\Theta}_a=\frac{\partial H}{\partial I_a}=\omega_a=\text{ constant}, \tag{2.89}$$

so that the trajectories are given by

$$\Theta_a = \omega_a t + \epsilon_a. \tag{2.90}$$

All that remains is to express the old coordinates q_a in terms of Θ_a by use of relations (2.79) and (2.80). In this case, we have

$$S(q_1, Q_2; f_1, f_2) = \int \sqrt{2\left(f_1 - \frac{1}{2}\omega_1^2 q_1^2\right)}\, dq_1 + \int \sqrt{2\left(f_2 - \frac{1}{2}\omega_2^2 q_2^2\right)}\, dq_2. \tag{2.91}$$

Because $\Theta_a = (\partial S/\partial I_a) = \omega_a(\partial S/\partial f_a)$, we have

$$\Theta_a = \omega_a \int \frac{dq_a}{\sqrt{2(f_a - \frac{1}{2}\omega_a^2 q_a^2)}} = \sin^{-1}\left(\frac{\omega_a q_a}{\sqrt{2f_a}}\right) \quad (a = 1, 2). \tag{2.92}$$

By using Eq. (2.90) we find that

$$q_a = \sqrt{\frac{2f_a}{\omega_a^2}} \sin(\omega_a t + \epsilon_a), \tag{2.93}$$

which solves the problem completely.

As a second example, let us consider the motion of a particle in two-dimensions under the action of a central potential. For the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r), \tag{2.94}$$

the two integrals of motion are

$$F_1 = H, \quad F_2 = p_\theta \ (=mr^2\dot{\theta}), \tag{2.95}$$

leading to the actions

$$I_2 = \frac{1}{2\pi}\oint p_\theta\, d\theta = \frac{F_2}{2\pi}\int_0^{2\pi} d\theta = F_2, \tag{2.96}$$

$$I_1 = \frac{1}{2\pi}\oint p_r\, dr = \frac{1}{\pi}\int \left\{2m\left[F_1 - \frac{I_2^2}{2mr^2} - V(r)\right]\right\}^{1/2} dr, \tag{2.97}$$

where the second integral is to be taken between the turning points in r. Given any $V(r)$, we can perform this integral and find I_1 in terms of I_2 and F_1. Inverting this relation, we can express the Hamiltonian (which is the same as F_1) in terms of I_1 and I_2. The rest of the analysis will proceed as described above. As a particular case, let us consider the motion under the potential

$$V(r) = -\frac{\alpha}{r} - \frac{\beta}{r^2}. \tag{2.98}$$

The two actions are

$$I_2 = F_2, \tag{2.99}$$

$$I_1 = \frac{1}{2\pi}\oint p_r\, dr = \frac{1}{2\pi}\oint \left(F_1 - \frac{F_2^2}{2mr^2} + \frac{\alpha}{r} + \frac{\beta}{r^2}\right)^{1/2} \sqrt{2m}\, dr$$

$$= -\sqrt{-2m\beta + F_2^2} + \alpha\left(\frac{m}{2|F_1|}\right)^{1/2}. \tag{2.100}$$

Solving for $H = F_1$ in terms of I_1, I_2, we get

$$H(I_1, I_2) = \frac{1}{2}\frac{m\alpha^2}{\left(I_1 + \sqrt{I_2^2 - 2m\beta}\right)^2}. \tag{2.101}$$

From the general theory, we know that the corresponding angular coordinates are linear functions of time: $\Theta_a = \omega_a t + \epsilon_a$, where the two frequencies are given by

$$\omega_1 = \frac{\partial H}{\partial I_1} = -\frac{m\alpha^2}{\left(I_1 + \sqrt{I_2^2 - 2m\beta}\right)^3}, \tag{2.102}$$

$$\omega_2 = \frac{\partial H}{\partial I_2} = -\frac{m\alpha^2}{\left(I_1 + \sqrt{I_2^2 - 2m\beta}\right)^3}\frac{I_2}{\sqrt{I_2^2 - 2m\beta}}. \tag{2.103}$$

We can easily express the old coordinates r, θ in terms of Θ_a as in the case of a two-dimensional oscillator and thus solve the problem completely.

Expressions (2.102) and (2.103) have some interesting features. We know that, in general, the motion will fill the surface of a two torus. An exception arises when the two frequencies are commensurable, i.e, their ratio is a rational number. This ratio is given by

$$\frac{\omega_1}{\omega_2} = \frac{\sqrt{I_2^2 - 2m\beta}}{I_2} = \sqrt{1 - \frac{2m\beta}{I_2^2}}. \tag{2.104}$$

If $\beta \neq 0$, then, this ratio can be rational only if $(1 - 2m\beta/I_2^2)^{1/2}$ is rational. If that happens, then the motion will be confined to a closed curve on the surface of the two torus, making the motion one dimensional rather than two dimensional. This reduction in the dimensionality of motion implies the existence of another isolating integral. To find this integral explicitly, we can proceed as in the case of the two-dimensional oscillator. By using Eq. (2.63) we can find the orbit of

the particle to be

$$\frac{1}{r} = \frac{\alpha m}{J^2}\left(1 - \frac{2\beta m}{J^2}\right)^{-1} + Q\cos\left[\left(1 - \frac{2\beta m}{J^2}\right)^{1/2}(\theta - \theta_0)\right], \quad (2.105)$$

where $Q \equiv [1 - (2m\beta/J^2)]^{-1}\{\alpha^2 m^2/J^4 + (2mE/J^2)[1 - (2m\beta/J^2)]\}^{1/2}$ and θ_0 is another constant. Solving for θ_0, we get

$$\theta_0 = \theta + \left(1 - \frac{2\beta m}{J^2}\right)^{-1/2}\cos^{-1}\left[\frac{1}{Q}\left\{\frac{1}{r} - \frac{\alpha m}{J^2}\left(1 - \frac{2\beta m}{J^2}\right)^{-1}\right\}\right]. \quad (2.106)$$

This quantity θ_0 is clearly an integral of the motion. If it is single valued, there will exist another isolating integral of motion, reducing the motion from two- to one-dimensional space. In general, $\cos^{-1} z$ will introduce a $2\pi n$ factor, giving

$$\begin{aligned}\theta &= \theta_0 - \left(1 - \frac{2\beta m}{J^2}\right)^{-1/2}\left[\mathrm{Cos}^{-1}\frac{1}{Q}\left\{\frac{1}{r} - \frac{\alpha m}{J^2}\left(1 - \frac{2\beta m}{J^2}\right)^{-1}\right\} \pm 2\pi n\right] \\ &= \theta_0 - \left(1 - \frac{2\beta m}{J^2}\right)^{-1/2}\mathrm{Cos}^{-1}\frac{1}{Q}\left\{\frac{1}{r} - \frac{\alpha m}{J^2}\left(1 - \frac{2\beta m}{J^2}\right)^{-1}\right\} \\ &\quad \pm 2\pi\left(1 - \frac{2\beta m}{J^2}\right)^{-1/2} n, \end{aligned} \quad (2.107)$$

where $\mathrm{Cos}^{-1} z$ (with an uppercase C) denotes the principal value. If $\beta \neq 0$, then θ can be single valued only if $(1 - 2\beta m/J^2)^{1/2}$ is a rational number. In this case we have an extra isolating integral.

The nature of the orbit in real space reflects the dimensionality of the motion in phase space. In general, the projection of phase-space motion (which fills a two-dimensional torus) will fill a two-dimensional region in the r, θ plane. But when the frequencies are commensurable, the trajectory spans a one-dimensional region in phase space and its projection will be a closed orbit in the r, θ plane.

The situation is more interesting in the case of $\beta = 0$, which corresponds to the Kepler problem. In this case, the two frequencies are identical and the motion always spans a one-dimensional region of the phase space; correspondingly, the orbit is a closed curve in real space. Once again, there must exist an additional integral of motion, which can be found along the following lines.

Consider a particle moving under a central force $\mathbf{f}(r) = -V'(r)\hat{\mathbf{r}}$. By direct differentiation, it is easy to see that

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{J}) = -mf(r)r^2\frac{d\hat{\mathbf{r}}}{dt}, \quad (2.108)$$

where $\mathbf{p}$, $\mathbf{J}$, and $\mathbf{r}$ are the momentum, the angular momentum, and the position of the particle. For the special case of inverse-square-law force, $\alpha \equiv f(r)r^2$ is a

constant and Eq. (2.108) implies that

$$\mathbf{A} \equiv \mathbf{p} \times \mathbf{J} - \frac{\alpha m}{r}\mathbf{r} \tag{2.109}$$

is conserved. This conserved vector, called the Runge–Lenz vector, has three components, but it satisfies the constraints

$$A^2 = 2mJ^2E + \alpha^2 m^2, \quad \mathbf{A} \cdot \mathbf{J} = 0. \tag{2.110}$$

The first one shows that the magnitude of **A** can be expressed in terms of other constants of motion; and the second one shows that **A** lies in the orbital plane. These two constraints reduce the number of independent components in **A** from three to one. This extra, independent constraint lowers the dimensionality of motion from two to one.

In this particular case, we can easily find the orbit of the particle by taking the dot product of **A** with **r** and by using the identity $\mathbf{r} \cdot (\mathbf{p} \times \mathbf{J}) = \mathbf{J} \cdot (\mathbf{r} \times \mathbf{p}) = J^2$. This gives

$$\mathbf{A} \cdot \mathbf{r} = Ar\cos\theta = J^2 - \alpha m r, \tag{2.111}$$

or, in more familiar form,

$$\frac{1}{r} = \frac{\alpha m}{J^2}\left(1 + \frac{A}{\alpha m}\cos\theta\right). \tag{2.112}$$

We see that **A** is in the direction of the major axis of the ellipse.

For future reference, the explicit solution for the Kepler problem in different cases is given here. By using the first equation of Eqs. (2.110), we can write the trajectory of Eq. (2.112) as

$$\frac{1}{r} = \frac{1}{a|(\epsilon^2 - 1)|}(1 + \epsilon\cos\theta); \quad (\epsilon^2 \neq 1), \tag{2.113}$$

where the semimajor axis a and the eccentricity ϵ of the ellipse are given by

$$a = \frac{\alpha}{2|E|}, \quad \epsilon^2 = 1 + \frac{2EJ^2}{m\alpha^2}. \tag{2.114}$$

The relation between the coordinate r and time t can be expressed in parametric form, depending on whether E is positive or negative:

$$\begin{aligned} r &= a(\epsilon\cosh\xi - 1), \quad t = \left(\frac{ma^3}{\alpha}\right)^{1/2}(\epsilon\sinh\xi - \xi) \quad (E > 0), \\ r &= a(1 - \epsilon\cos\xi), \quad t = \left(\frac{ma^3}{\alpha}\right)^{1/2}(\xi - \epsilon\sin\xi) \quad (E < 0). \end{aligned} \tag{2.115}$$

The above examples clearly illustrate that, if the system is integrable, then the motion takes place on an N-dimensional torus and – in general – fills the surface. The exceptional situation arises when the ratio between any two of the

frequencies is a rational number. In such a case, the motion is confined to a space with a lower dimension than N. When such a situation arises, there will exist another isolating integral of motion. In the case of the two-dimensional harmonic oscillator, this is given by Eq. (2.75); for the Kepler problem, it is the Runge–Lenz vector.

Exercise 2.13

Hodograph for Kepler problem: The curve traced by the tip of the velocity vector of a particle moving under some force law is called a hodograph. Show that the hodograph of a particle moving in the Kepler problem is a circle in the case of bounded motion and part of a circle in the case of unbounded motion.

Exercise 2.14

Kepler motion of a binary system: Consider two stars of masses m_1 and m_2 orbiting around the common centre of mass (in an elliptical path) under their mutual gravitational attraction. The plane of the orbit is inclined at an angle i to the line of sight connecting the observer to the focus. From the Doppler shift of the radiation emitted by star 1, it is possible to determine the velocity of that star along the line of sight. Show how this information will help us to determine the combination

$$C = \frac{m_2^3 \sin^3 i}{(m_1 + m_2)^2}. \tag{2.116}$$

This procedure is of importance in determining the mass of an invisible companion in a binary stellar system.

2.6 Adiabatic Invariance

The Hamiltonian of any physical system depends on some of the parameters that characterise the system and are treated as constants during the time evolution. For example, the frequencies ω_a of a two-dimensional oscillator or the mass of the Sun in planetary motion belong to this set. There are physical situations in which such parameters may change slowly with time rather than remain strictly constant (for example, the mass of the Sun may slowly decrease because of solar wind). We would like to understand how the slow change in the parameters of the problem affects our solution. It turns out that the formalism developed above – in terms of action-angle variables – comes in handy in this case.

To illustrate this idea let us consider a one-dimensional system executing a finite motion with period T. Let λ be a parameter in the Hamiltonian that varies slowly with time in the sense that $T(d\lambda/dt) \ll 1$. If λ were strictly constant, the energy E of the system would be conserved; when λ varies slowly, E would not be conserved but we expect the rate of change of energy with time $\dot{E}$ to be small. If we now average this rate over a period T, the rapid variations will be averaged out and the resulting value of $\langle \dot{E} \rangle$ will give the secular slow change in E that is

due to the variation $\dot{\lambda}$. Because the variation of λ leads to an average variation of E, we can express the averaged E as some function of λ; equivalently, we can express this relation as the constancy of some function of E and λ. Such functions, which remain constant when the parameters of the system are varied slowly, are called adiabatic invariants. We now show that the actions I_a are adiabatic invariants and remain unchanged when the parameters of the system are varied slowly.

Let $H[q, p, \lambda(t)]$ be the Hamiltonian of the system that depends on the slowly varying parameter λ. The rate of change of energy of the system is

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} = \frac{\partial H}{\partial \lambda}\dot{\lambda}. \tag{2.117}$$

On the right-hand side, $\dot{\lambda}$ is a slowly varying quantity but $(\partial H/\partial\lambda)$ depends on the rapidly varying quantities q and p. To determine the steady, secular change in E we average this expression over the period of motion. Because $\dot{\lambda}$ varies slowly, we can take it outside the averaging and write

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{d\lambda}{dt}\left\langle \frac{\partial H}{\partial \lambda} \right\rangle, \tag{2.118}$$

where we treat λ as a constant in the function $(\partial H/\partial\lambda)$ that is being averaged. Expressing the time average explicitly as

$$\left\langle \frac{\partial H}{\partial \lambda} \right\rangle = \frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda}\, dt \tag{2.119}$$

and transforming from dt to dq by $dt = dq/[(\partial H/\partial p)]$, we find that

$$\left\langle \frac{dE}{dt} \right\rangle = \left(\frac{d\lambda}{dt}\right)\frac{\oint (\partial H/\partial\lambda)\, dq/(\partial H/\partial p)}{\oint dq/(\partial H/\partial p)}. \tag{2.120}$$

The circle on the integral sign denotes that the integration is taken over the complete range of variation of the coordinates during one period. Because these integrals are taken along paths in which the Hamiltonian has a constant value E, we can express the momentum as a definite function of the variable coordinate q and the two constant parameters E and λ. Setting $p = p(q; E, \lambda)$ and differentiating the equation $H(q, p, \lambda) = E$ with respect to λ, we get $(\partial H/\partial\lambda) = -[(\partial H/\partial p)][(\partial p/\partial\lambda)]$ or

$$\frac{(\partial H/\partial\lambda)}{(\partial H/\partial p)} = -\frac{\partial p}{\partial \lambda}. \tag{2.121}$$

By using this in Eq. (2.120) and writing the integrand of the denominator as

$(\partial p/\partial E)$, we get

$$\left\langle \frac{dE}{dt} \right\rangle = -\left(\frac{d\lambda}{dt} \right) \frac{\oint (\partial p/\partial \lambda)\, dq}{\oint (\partial p/\partial E)\, dq}, \tag{2.122}$$

that is,

$$\oint \left(\frac{\partial p}{\partial E} \left\langle \frac{dE}{dt} \right\rangle + \frac{\partial p}{\partial \lambda} \frac{d\lambda}{dt} \right) dq = 0. \tag{2.123}$$

This is equivalent to the condition $\langle dI/dt \rangle = 0$, where

$$I(E, \lambda) \equiv \oint p(q; E, \lambda) \frac{dq}{2\pi}, \tag{2.124}$$

and the integral is taken over a path with given E and λ. This result can be directly generalised to systems with more degrees of freedom and shows that the actions introduced by Eq. (2.83) are adiabatic invariants.

Note that adiabatic invariance of actions requires that the change in the parameter be small during a complete cycle of motion, that is, we must have $T\dot{\lambda} \ll 1$, where T is the period. We can imagine situations in which this condition is violated even though the external parameter changes slowly in some other prespecified manner. Adiabatic invariance will not be a valid approximation under such circumstances.

As a first example of adiabatic invariance, consider a harmonic oscillator whose frequency is varying slowly. Because the action is E/ω, we can conclude that the energy changes in proportion to the frequency.

A more interesting example is given by the Kepler problem. In this case, the orbit is an ellipse with the equation

$$\frac{l}{r} = 1 + \epsilon \cos\theta, \quad l = \frac{J^2}{m\alpha}, \quad \epsilon = \sqrt{1 + \left(\frac{2EJ^2}{m\alpha^2} \right)}, \tag{2.125}$$

where $2l \equiv 2a(\epsilon^2 - 1)$ is the latus rectum, ϵ is the eccentricity [see Eq. (2.113)], and J and E are the conserved angular momentum and energy, respectively. The actions in this case are

$$I_2 = J, \quad I_1 = -J + \alpha \left(\frac{m}{2|E|} \right)^{1/2} \tag{2.126}$$

[see Eqs. (2.99) and (2.100) with $\beta = 0$], it is easy to see that

$$l = \frac{I_2^2}{m\alpha}, \quad \epsilon^2 = 1 - \left(\frac{I_2}{I_2 + I_1} \right)^2. \tag{2.127}$$

Suppose now that the parameter m or α changes slowly with time. Because the actions I_1 and I_2 remain invariant, the eccentricity of the orbit does not change. On the other hand, the scale of the orbit determined by l changes inversely

as $m\alpha$. These conclusions, which are straightforward to obtain from action-angle variables and adiabatic invariance, are not so easy to derive from more direct approaches.

2.7 Perturbation Theory for Nonintegrable Systems

For a closed system with $2N$-dimensional phase space, the only obvious constant of motion is the Hamiltonian. This restricts the motion of the system to a $(2N-1)$ submanifold. On the other hand, for a fully integrable system, the motion is confined to an N-dimensional submanifold, which is of lower dimensionality compared with $(2N-1)$ except in the trivial case of $N=1$. If the system is made of several particles, each moving in three-dimensional physical space, then homogeneity of time and space and isotropy of space will lead to the existence of a total of seven integrals of motion corresponding to the conservation of energy, three components of linear momentum, and three components of angular momentum. For a system with k particles, the phase space is $6k$ dimensional and the existence of the above conserved quantities confines the motion to a $(6k-7)$ dimensional space. If the system is integrable, then the motion will be confined to $3k$-dimensional space, which is of lower dimensionality for all $k > 2$. It is clear that integrability puts a severe restriction on the region of phase space that the dynamical system can explore.

This prompts the following question: Is integrability a rule or an exception? In other words, do all Hamiltonian dynamical systems in $2N$-dimensional phase space possess N integrals of motion? The question is nontrivial because it is possible to construct systems for which fairly complicated integrals of motion can exist. For example, in the case of the Kepler problem, it is impossible to decide by inspection the conservation of the Runge–Lenz vector. So we could take the point of view that all systems are integrable and our inability to find N integrals of motion is only due to lack of mathematical expertise. On the other hand, we could also take the point of view that the class of integrable Hamiltonian is a very small subset of the class of all possible Hamiltonians. In that case, the smallest perturbation to an integrable system will destroy the structure of the tori, and the system will – in due course of time – fill a region of phase space of higher dimensionality than $\mathcal{M}$.

These issues are of importance in the study of several astrophysical systems, especially in the study of the long-term evolution of planetary systems. A planet moving under the inverse-square-law force of the Sun constitutes an integrable system, and its trajectory in phase space is confined to a torus. Even if the force law were changed by the addition of an inverse cube term, say, the system would still be integrable. (In the first case, the trajectory is closed on the torus, whereas, in the second case, it will fill the torus.) However, from the point of view of celestial mechanics, it is important to know what happens to the trajectory when the perturbations from the other planets are taken into account. The system can

no longer be described by a central potential and hence we cannot, by inspection, obtain N integrals of motion. We would like to know, in this case, whether the perturbation destroys the tori completely or whether they survive.

Detailed investigations have shown that most of the tori persist under perturbations, although in a distorted form. Some of them are destroyed (and these do not form a set of measure zero), with the measure of destruction growing with the perturbation. The destroyed tori are distributed among those that are preserved in a very pathological manner. These assertions were rigorously proved in what is now called the Kolmogorov–Arnold–Moser (KAM) theorem. Some features of this analysis are now described.

Let us consider the effect of a perturbation on an integrable system by assuming that the total Hamiltonian is given by

$$H(I, \Theta) = H_0(I) + \epsilon H_1(I, \Theta). \tag{2.128}$$

Here $H_0(I)$ is the Hamiltonian corresponding to an integrable system and I, Θ are the action-angle variables for this system. [Both I and Θ denote N tuplets (I_a, Θ_a) with $a = 1, 2, \ldots, N$. We suppress the subscripts when no confusion is likely to arise.] The second term $\epsilon H_1(I, \Theta)$ is a perturbation characterized by the strength ϵ. Because the total Hamiltonian depends on Θ (through the perturbation), I and Θ are no longer action-angle variables for the whole system, although they are perfectly good canonical coordinates. If tori exist for the perturbed system, then there must exist a new set of action-angle variables I', Θ' such that

$$H(I, \Theta) = H'(I'), \tag{2.129}$$

and the new variables must be related to the old by a canonical transformation produced by some generating function $S(\Theta, I')$, with

$$I_a = \frac{\partial S}{\partial \Theta_a}, \quad \Theta'_a = \frac{\partial S}{\partial I'_a}. \tag{2.130}$$

Substituting these relations into Eq. (2.129), we get the condition that S must satisfy:

$$H[\nabla_\Theta S(\Theta, I'), \Theta] = H'(I'). \tag{2.131}$$

Whether the tori exist or not is thus reduced to the problem of finding acceptable solutions to this equation.

Let us examine whether a solution can be found as a power series in the perturbation parameter ϵ. The zeroth-order term must be $S_0 = \Theta_a I'_a$, which generates the identity transformation. Therefore, we write

$$S = \Theta_a I'_a + \epsilon S_1(\Theta, I') + \cdots. \tag{2.132}$$

Substituting this expression in Eq. (2.131), we get

$$H_0(I' + \epsilon \nabla_\Theta S_1 + \cdots) + \epsilon H_1(I' + \cdots, \Theta) = H'(I'), \tag{2.133}$$

with the notation $\nabla_\Theta S = (\partial S/\partial \Theta_a)$, etc. To first order in ϵ,

$$H_0(I') + \epsilon[\nabla_{I'} H_0(I') \cdot \nabla_\Theta S_1 + H_1(I', \Theta)] = H'(I'). \tag{2.134}$$

We now note that $\omega_{0a}(I') = [\partial H_0(I')/\partial I'_a]$ is the frequency of the unperturbed motion and that H_1 and S_1 can be expanded as periodic functions in Θ:

$$\begin{aligned} H_1(I', \Theta) &= \sum_{\mathbf{m}} H_{1\mathbf{m}}(I') \exp(i\mathbf{m} \cdot \Theta), \\ S_1(\Theta, I') &= \sum_{\mathbf{m} \neq 0} S_{1\mathbf{m}}(I') \exp(i\mathbf{m} \cdot \Theta). \end{aligned} \tag{2.135}$$

(Note that H_1 and S_1 are functions of q and p, which in turn are periodic functions of Θ.) Substituting Eq. (2.135) into Eq. (2.134) and equating the Fourier coefficients, we get

$$H'(I') = H_0(I') + \epsilon H_{10}(I') + \cdots \quad (\mathbf{m} = 0), \tag{2.136}$$

$$S_{1\mathbf{m}}(I') = +\frac{i H_{1\mathbf{m}}(I')}{m_a \omega_{0a}(I')} + \cdots \quad (\mathbf{m} \neq 0). \tag{2.137}$$

Thus, to the lowest order, the generator of the new tori is given by

$$S(\Theta, I') = \Theta \cdot I' + i\epsilon \sum_{\mathbf{m} \neq 0} \frac{H_{1\mathbf{m}}(I')}{m_a \omega_{0a}(I')} \exp(i\mathbf{m} \cdot \Theta) + \cdots. \tag{2.138}$$

It might appear now that we have achieved what we originally wanted; namely, to the lowest order in perturbation theory, we have a generating function that will lead to a new set of action-angle variables for the full system and hence to a new set of tori. By repeating this process order by order in ϵ, we will find the full generating function and the tori, thereby showing that perturbations leave an integrable system still integrable.

These conclusions, however, are wrong. The key difficulty arises from the denominators in Eq. (2.137) that have the terms $m_a \omega_{0a}$. Whenever these quantities vanish, the perturbation theory breaks down. This situation arises when the frequencies of motion on the unperturbed torus are commensurable – that is, if the original orbits are closed and the fundamental frequencies are in resonance. In fact, even when the frequencies are incommensurable, it is always possible to find a set of m_a such that $m_a \omega_{0a}$ is arbitrarily small. This raises the doubt as to whether the series ever converges. Note that we have to worry about both the convergence of the sum over individual terms as well as the convergence of the series in powers of ϵ.

In the study of planetary systems etc., the unperturbed motion usually does not have commensurable frequencies (although there are some very important and interesting exceptions in the solar system, as we shall see in Vol. II). In that case, the trouble arises for only large values of m_a. For such large values of m_a the corresponding Fourier coefficients $H_{1\mathbf{m}}$ are very small. If we truncate the sums

before these large values of m_a come into play and work with a few low-order terms, we can predict the motion for a long time. The success of these predictions is important from the practical point of view of celestial mechanics, but does not help in answering the key theoretical question of *arbitrarily* late-time behaviour of the system.

Significant progress in this problem can be made if the nature of the perturbative analysis is changed. Instead of expanding in a power series about the original Hamiltonian, we can do a perturbation series with the nth term defined as a perturbation around the system defined by all terms up to the $(n-1)$th term. From such an analysis, we can prove the following result: The process of generating (perturbed) tori does converge almost always for a small but finite ϵ. So most of the trajectories stay on an N-dimensional manifold for all times and do not explore the $2N-1$ dimensional energy surface. More precisely, all initial tori, whose frequency ratios are sufficiently irrational, are preserved. The condition for sufficient irrationality may be stated (in the case of $N=2$, for example) as the following condition on the ratio of the two frequencies ω_{01} and ω_{02}:

$$\left|\frac{\omega_{01}}{\omega_{02}} - \frac{r}{s}\right| > \frac{K(\epsilon)}{s^{2.5}} \tag{2.139}$$

for all integers r and s, where K is a number that is independent of r and s and tends to zero as ϵ tends to zero. Note that this condition says the frequency ratio should be sufficiently far away from any rational number r/s. The tori that are excluded by this condition are the ones that are likely to be seriously affected by the perturbation.

Let us estimate the measure of these tori. Taking $0 \leq (\omega_{01}/\omega_{02}) \leq 1$ (without any loss of generality), we remove from the interval [0, 1] all patches that violate condition (2.139), i.e., all patches that satisfy

$$\left|\frac{\omega_{01}}{\omega_{02}} - \frac{r}{s}\right| < \frac{K(\epsilon)}{s^{2.5}}. \tag{2.140}$$

Thus we take off a region of length $(K/s^{2.5})$ around each rational r/s in the interval [0, 1]. This amounts to taking away a total length that is less than

$$\sum_{s=1}^{\infty} \frac{K}{s^{2.5}} s = K \sum_{s=1}^{\infty} \frac{1}{s^{1.5}} = K\zeta(1.5) \cong 2.6K, \tag{2.141}$$

which tends to zero as the perturbation gets smaller. [The factor s in Eq. (2.141) denotes the number of r values within r/s on 0 to 1; actually, the result in Eq. (2.141) is an overestimate because we have double counted neighbourhoods which overlap.] Clearly, we are left with a set of nonzero measure, even after we remove the tori that are close to rationals.

What happens to these tori, which originally had commensurable [or nearly commensurable, in the sense of condition (2.140)] frequencies? Numerical

investigations suggest that these tori are destroyed and the orbits in them are redistributed in phase space. The low-order resonances in the original system are almost always destroyed by the perturbation (and can lead to interesting physical effects, which we will study in Vol. II), although the fate of higher-order resonances is difficult to study.

2.8 Surface of Section

It is clear from the discussion in Section 2.7 that the nature of motion in the phase space for nonintegrable systems can be extremely complex. In principle, we can attempt to study such a system by numerically integrating the equations of motion starting from some initial condition. Although this approach is of great utility (and is often resorted to) in understanding the behaviour of specific systems, it does not, by itself, help us to understand the general characteristics of motion. To do so, it is convenient to represent the key features of the motion by use of a technique called surface of section.

To illustrate this technique, let us consider the motion of a particle in the $x-y$ plane under the action of a potential $\phi(x, y)$. The phase space for this system is four dimensional. Conservation of energy will restrict the motion to a three-dimensional surface in phase space. The path of the system $\mathcal{C}(t)$ on this three-dimensional surface will cut the $y=0$ plane in a two-dimensional surface, which can be characterised by the two coordinates x, and p_x. We can obtain valuable insight into the nature of the motion by plotting the intersection of the curve $\mathcal{C}(t)$ with the $y=0$ surface. In general, the motion will be spread over a bounded region in the x, p_x plane, as we expect the motion to span a two-dimensional surface. If the system has any other hidden integrals of motion, then the motion can become one dimensional and the points of intersection of $\mathcal{C}(t)$ with the $y=0$ surface will lie on a curve.

For a system with $N=2$, the phase space is four dimensional with the coordinates x, y, p_x, and p_y. Conservation of energy restricts the motion to a three-dimensional surface defined by

$$\frac{1}{2}(p_x^2 + p_y^2) + \phi(x, y) = E = \text{constant}. \tag{2.142}$$

(We shall take $m=1$.) The intersection of this three-dimensional surface with the surface $y=0$ defines the surface of section for the dynamical problem, on which any point has the coordinates x, and p_x. Given (x, p_x) and the condition $y=0$, we can determine p_y as

$$p_y = +\{2[E - \phi(x, 0)] - p_x^2\}^{1/2}, \tag{2.143}$$

where we have chosen the positive square root by convention. Given the energy E and suitable initial conditions, the equations of motion can be integrated to give the orbit of the particle. This orbit will repeatedly cut the surface of section

at different values of x and p_x. If there are no other integrals of motion, we expect these points to be scattered in a two-dimensional region bounded by the curve

$$\frac{1}{2}p_x^2 + \phi(x, 0) \leq E. \tag{2.144}$$

In case the system has some other hidden integrals of motion, the orbit will be confined to a lower-dimensional surface in phase space and hence will form a smooth curve in the xp_x plane. In general, very different kinds of surface of section are generated by different Hamiltonians. Even for a given Hamiltonian, the nature of the points (induced by the motion) on the surface of section can be widely different for different initial conditions.

To illustrate these concepts, we consider the motion of three-point particles under their mutual gravitational attraction subject to the approximations discussed in Section 2.3. As in that section, it is convenient to use a system of axes (x, y) that corotates with m_1 and m_2, with the x axis pointing toward m_2, say. In this frame m_1 and m_2 are stationary with fixed coordinates $(-\mu, 0)$ and $(1-\mu, 0)$, respectively, where $\mu = m_2/(m_1 + m_2)$. If the coordinates of m_3 are (x, y) then the equations of motion are given by Eqs. (2.35). This system can be brought into a Hamiltonian form with 2 degrees of freedom by choosing

$$q_1 = x, \quad q_2 = y, \quad p_1 = \dot{x} - y, \quad p_2 = \dot{y} + x, \tag{2.145}$$

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + p_1 q_2 - p_2 q_1 - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}, \tag{2.146}$$

with ρ_1 and ρ_2 defined as in Eq. (2.37). It is obvious that the Hamiltonian is an integral of motion; it is conventional to define it in terms of a quantity called the Jacobi integral, which is given by $C = -2H$. In terms of the original variables, C is given by

$$C = x^2 + y^2 + \frac{2(1-\mu)}{\rho_1} + \frac{2\mu}{\rho_2} - \dot{x}^2 - \dot{y}^2. \tag{2.147}$$

Here C differs from that in Eq. (2.37) by only a constant $\mu(1-\mu)$.

No other integral for this system is known, and hence the trajectories have to be computed by numerical integration. Let us consider the surface of section defined by $y = 0$, $\dot{y} > 0$ for the case of equal masses ($\mu = 1/2$). Figure 2.3 shows the surface of section for $C = 4.5$. It appears as though the system is nearly integrable and all accessible regions are covered by well-connected regular curves. The orbit itself shows great regularity, with m_3 staying comparatively close to m_1.

When the value of C is lowered slightly to $C = 4$, the surface of section changes to that shown in Fig. 2.4. Now the curves do not fill the accessible region in a regular fashion. All the isolated points correspond to a single trajectory that has a chaotic character. For a still lower value of $C = 3.5$ (see Fig. 2.5) the chaotic region increases in extent. The orbit has a very disordered character. In a typical

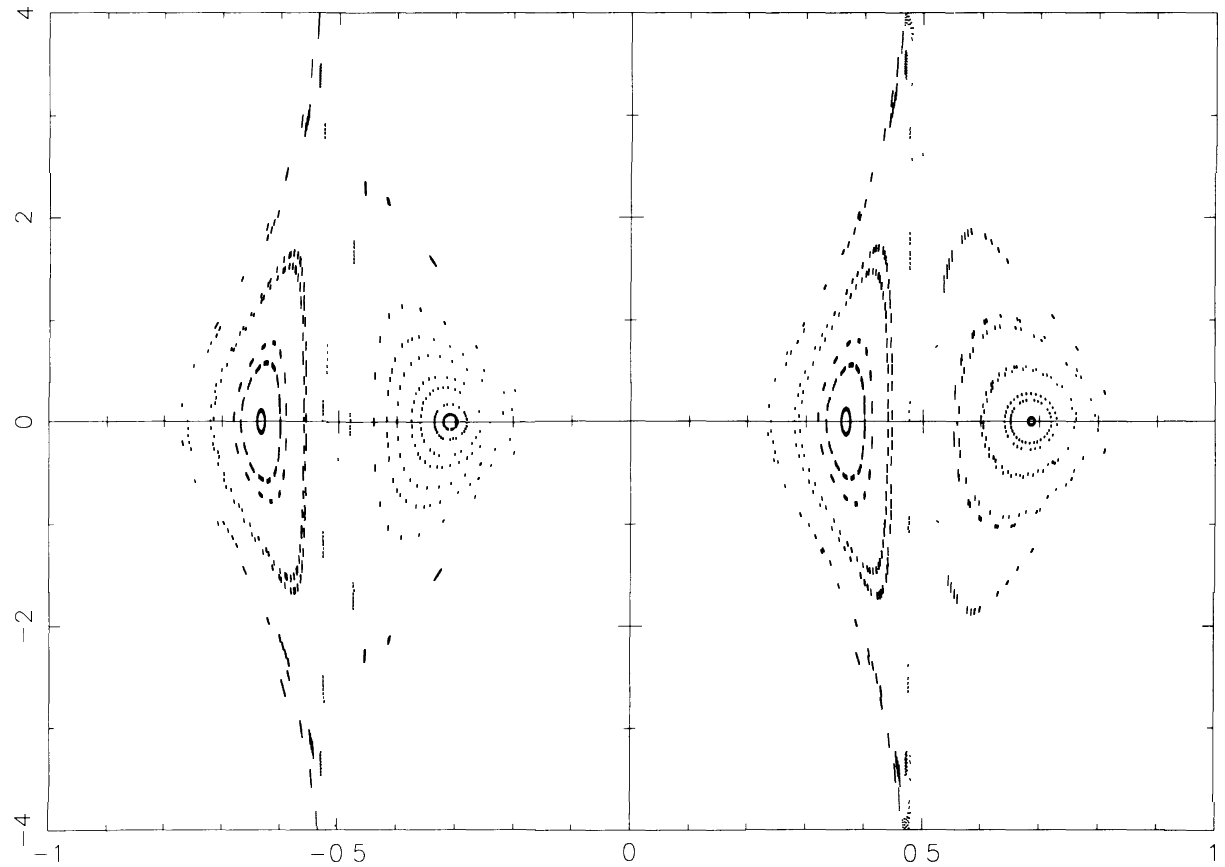

Fig. 2.3. Surface of section for restricted three-body problem for $C = 4.5$.

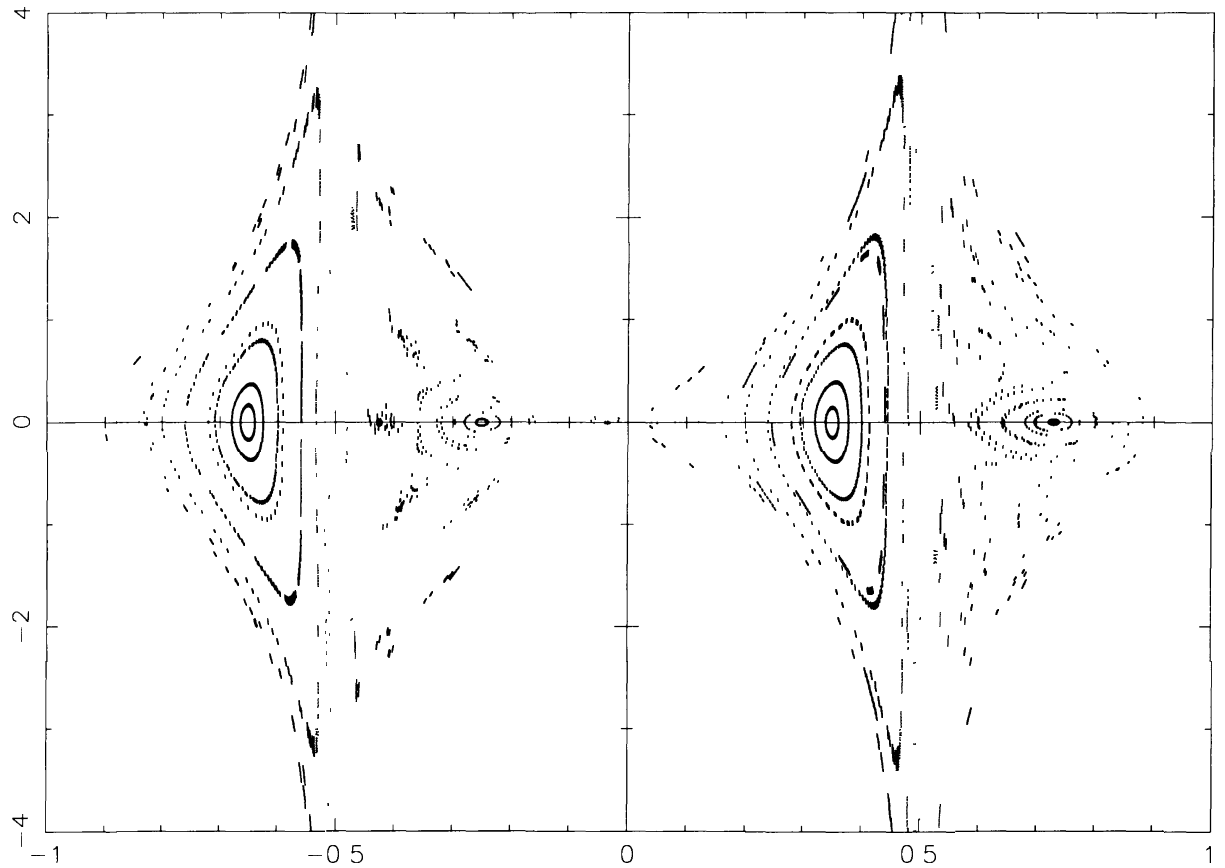

Fig. 2.4. Surface of section for restricted three-body problem for $C = 4$.

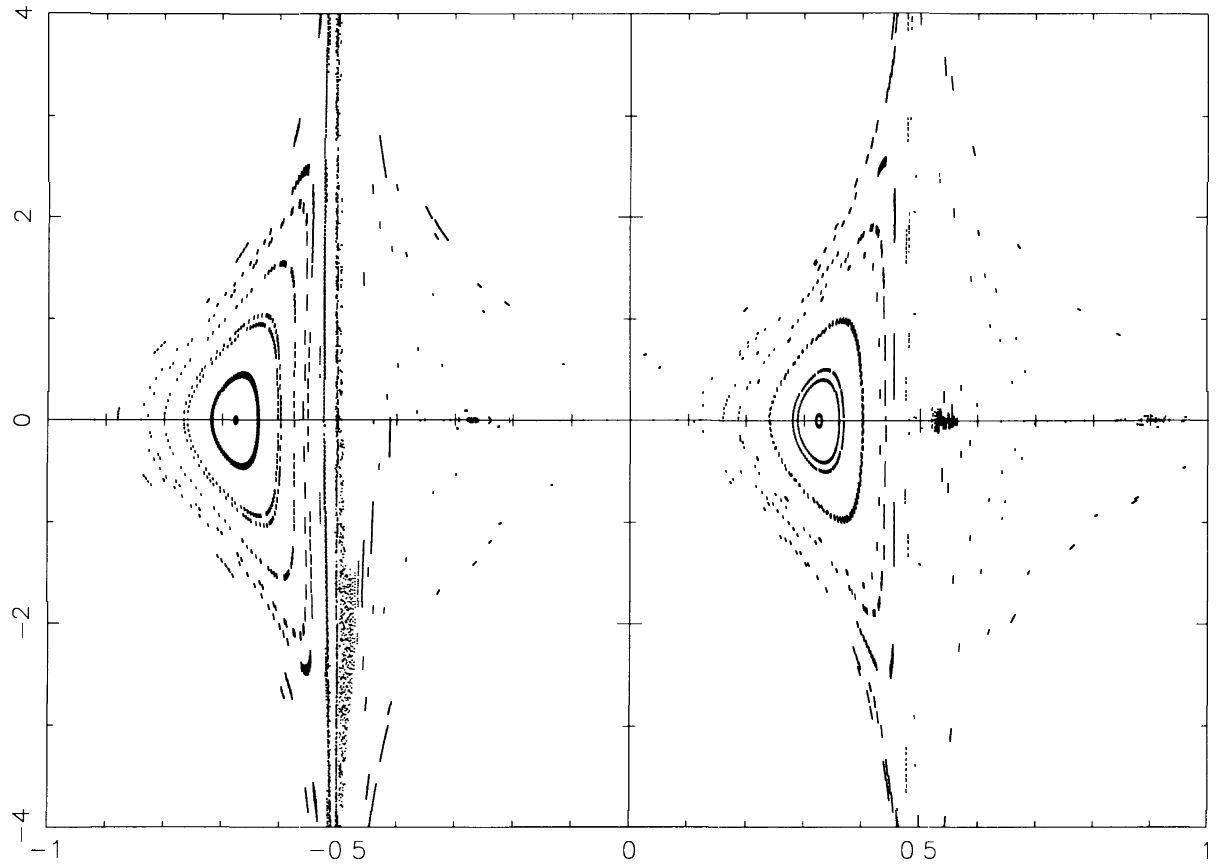

Fig. 2.5. Surface of section for restricted three-body problem for $C = 3.5$.

motion, m_3 will describe a few loops around m_1, then jump to m_2, then back to m_1, etc.

This example shows clearly that the dynamical evolution of even simple looking systems can be highly irregular. What is more, the behaviour of the Poincaré section changes drastically when some parameters in the problem change slightly. The motion of the particle as depicted in the Poincaré section changes from a fairly regular form to an aperiodic, chaotic form when the parameters of the problem change.

Some aspects of this transition to chaos can be understood along the following lines. The points in the surface of section are generated when the equations of motion are integrated for a particle. Mathematically, we can model such a system by giving a map of the form $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$, where $\mathbf{x}_n$ gives the coordinates of the nth point in the surface of section and the function f is a rule that allows us to determine the $(n + 1)$th point from the nth point. This suggests that we should study the evolution of the coordinates under iterative maps in order to understand the dynamics of complex systems. Simplifying the model still further, let us consider one-dimensional maps of the type $x_{n+1} = f(x_n, \mu)$, where μ is some parameter in the problem. Given a particular value of μ, the form of the function $f(x, \mu)$, and a starting value x_0, this rule will generate a succession of points $x_0, x_1, x_2 \ldots$. We are interested in studying the general features of this sequence.

To begin with, it is clear that the sequence will repeat itself with a period k if $x_{n+k-1} = f(x_n, \mu)$. Given the functional form of f, we could examine whether

such nontrivial solutions exist for specific values of x and μ. Further, if $k=1$, that is, if $x_n = f(x_n, \mu)$, such a value of x_n is called a fixed point of the map. The structure of mappings can usually be understood in terms of the fixed points and the periods in the sequences.

As an example, consider the mapping generated by the function $f(z, \mu) = 4\mu z(1-z)$, where μ is a parameter in the range $0 \leq \mu \leq 1$. This function maps the interval (0, 1) to itself. The fixed points of this map are the roots of the equation $z = 4\mu z(1-z)$ and are given by (1) $z=0$ and (2) $z = [1-(4\mu)^{-1}]$. The first solution is trivial and exists for all values of μ, whereas the second solution exists in the relevant interval (0,1) only if $\mu > (1/4)$.

These fixed points are the solutions to the equation $z = f(z, \mu)$. Let us consider the effect of varying the value of x slightly around the fixed point. We do this by writing $x_n = (z + \xi_n)$ and expanding the map in a Taylor series in ξ. This leads to the relation $\xi_{n+1} = f'(z, \mu)\xi_n$, where the prime denotes the first derivative with respect to z. This equation admits solutions of the form $\xi_n = c[f'(z, \mu)]^n$, where c is a constant. The perturbation ξ_n will decay with successive iterations if and only if $|f'(z, \mu)| < 1$. Such a fixed point is called stable. For the map discussed above, the nontrivial fixed point in (2) is stable if $\mu < (3/4)$. Thus in the range $(1/4) < \mu < (3/4)$ there exists a nontrivial, stable fixed point for our mapping at $z = [1-(4\mu)^{-1}]$. Figure 2.6(a) shows a geometrical way of identifying the successive values of the coordinate for a particular value of μ in this range. The thick curve is described by the equation for our map $y = 4\mu x(1-x)$. We start with a particular value x_0 in the x axis and identify the point to which x_0 is mapped by drawing a vertical line at $x = x_0$ and finding the point of intersection with the thick curve. The diagonal dashed line corresponds to $y = x$; hence a line parallel to x axis from the point of intersection will cut this diagonal line at an x coordinate that corresponds to $x_1 = 4\mu x_0(1-x_0)$. The procedure can be now repeated from x_1 to find x_2, etc. It is clear that the intersection of the dashed line and the thick curve will correspond to a fixed point. The evolution proceeds towards the fixed point x_{11} and winds around it.

When the value of μ is beyond 3/4, a new feature emerges in the evolution [see Fig. 2.6(b)]. A new solution with period 2 takes over from the original stable fixed point. The sequence now oscillates between two values, x_{2-} and x_{2+}. These values are determined by the simultaneous solutions of the equations $x_{2+} = 4\mu x_{2-}(1-x_{2-})$ and $x_{2-} = 4\mu x_{2+}(1-x_{2+})$. The solution is given by

$$x_{2\pm} = \frac{1}{8\mu}\left[1 + 4\mu \pm (16\mu^2 - 8\mu - 3)^{1/2}\right]. \tag{2.148}$$

This behaviour persists as long as $\mu < (1+\sqrt{6})/4 \cong 0.86$. When μ is increased above this value, another new solution appears that has a period of 4. This pattern of evolution goes on ad infinitum; there is a series of critical values $\mu = 0.75, 0.86, \ldots$, such that as μ crosses these values the period of the oscillation doubles.

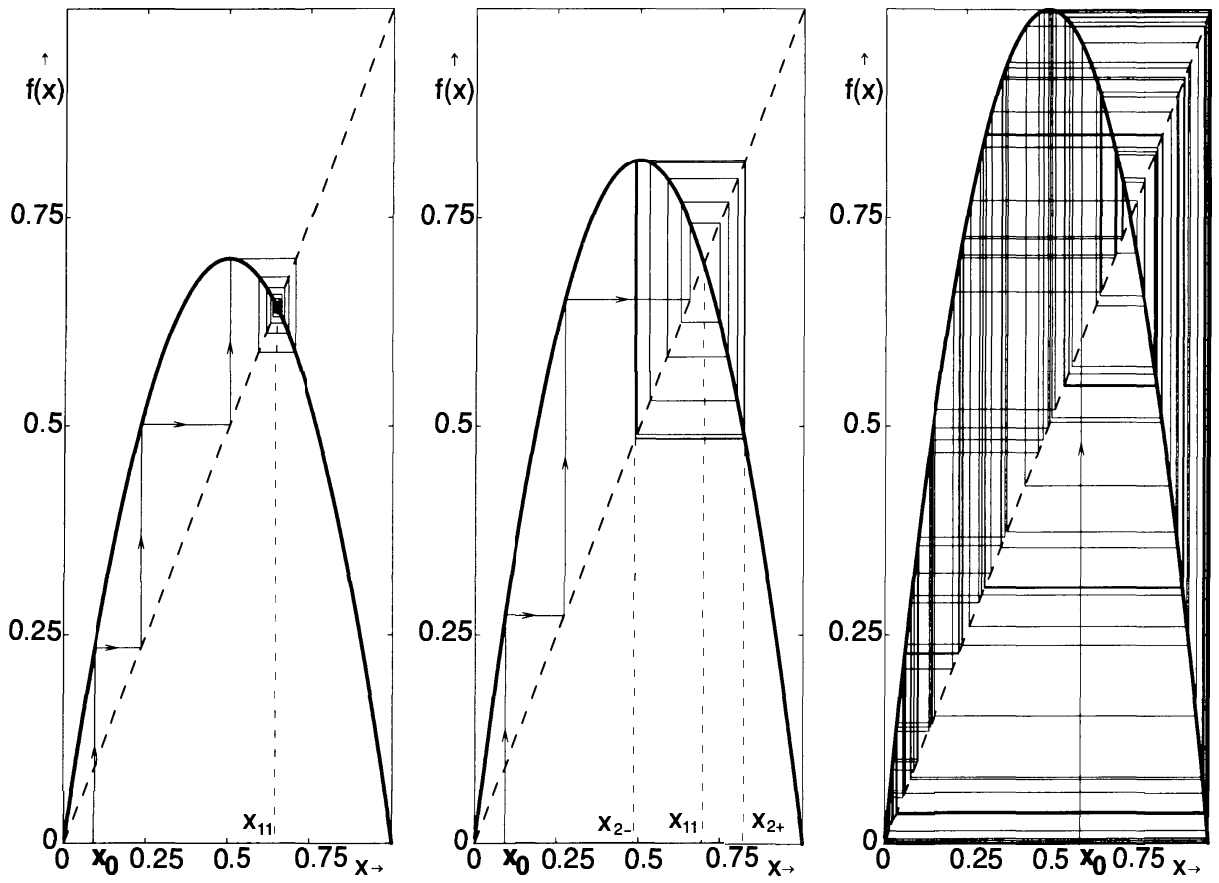

Fig. 2.6. Geometrical construction to determine successive values of the logistic map: (a) $\mu = 0.7$ (b) $\mu = 0.775$, (c) $\mu = 1$.

Finally, let us consider the maximum allowed value: $\mu = 1$. In this case, the sequence from a generic starting value will lead to a complex, aperiodic behaviour, shown in Fig. 2.6(c). The actual terms in the sequence depend sensitively on the starting value, but the overall nature of the evolution remains the same.

This example illustrates that very complex evolution can be exhibited by a seemingly simple system. It also shows that period doubling at critical values of the parameter can lead to chaotic behaviour eventually. Because the dynamical system considered in Eq. (2.146) (the reduced three-body problem) is at least as complicated as this example, it is understandable that the sequence of points in the Poincaré section shows complex behaviour and is sensitive to the value of the input parameter.

Exercise 2.15

Lorenz equations: One of the early discoveries of chaos was in a model for atmospheric convection that was governed by three variables x, y, and z evolving according to the equations $\dot{x} = 10(y - x)$, $\dot{y} = 28x - xz - y$, and $\dot{z} = xy - (8z/3)$. Integrate these equations numerically for different initial conditions and show that x, y, and z follow nonrepeating orbits in the phase space that can be chaotic.